MBA 2305
BUSINESS MATHEMATICS

স্কুল অব বিজনেস
SCHOOL OF BUSINESS

Bangladesh Open University
বাংলাদেশ উন্মুক্ত বিশ্ববিদ্যালয়

স্কুল অব বিজনেস

SCHOOL OF BUSINESS

বাংলাদেশ উন্মুক্ত বিশ্ববিদ্যালয়

MBA 2305
Business Mathematics

Course Development Team

Writer

Professor Dr. M. A. Taher
Department of Management
University of Chittagong
Chittagong

Dr. Mohammed Shamim Uddin Khan
Associate Professor
Department of Finance & Banking
University of Chittagong
Chittagong

Editor and Style Editor

S. M. Miraj Ahmmod
Associate Professor
School of Business
Bangladesh Open University

Coordinator

Prof. Dr. M. Ekramul Hoque
School of Business
Bangladesh Open University

This book has been published after being refereed for the students of School of Business, Bangladesh Open University

MBA 2305
BUSINESS MATHEMATICS

Published by: Publication, Printing and Distribution Department, Bangladesh Open University, Gazipur - 1705. **Date of Publication: July, 2008. Computer Compose & Desk-Top Processing:** Mohammad Wahiduzzaman Howlader, **Cover Graphics:** Abdul Malek, **Cover Design:** Monirul Islam, **Printed by: ....**

PREFACE

This text book titled ***Business Mathematics*** is designed and developed for the MBA students of Bangladesh Open University. It is written in modular form and is the first of its kind on Business Mathematics in Bangladesh. The lessons have been so designed that learners find them easy to understand.

The book has twelve units comprising 44 lessons. We do not claim it to be an original contribution. Rather it should be regarded as a textbook of ideas from various renowned writers in Business Mathematics. We have also quoted from different textbooks on Mathematics usually followed by post-graduate students in our universities. Our endeavour has been to present the lessons in a very lucid manner so that they can be understood and assimilated by an average distance learner of the MBA program within the stipulated period of a semester.

Each unit is almost equivalent to one chapter of a conventional text book and contains two to six lessons. Each of them starts with unit “highlights”. In fact the lessons are like the lecture notes of a classroom teacher, each starts with “lesson objectives” and ends with “review questions”. The review questions include practice problems and multiple choice questions (MCQs). We hope that self learners will not find much difficulty in understanding the lessons by themselves and will need only a little help from the tutor.

We are grateful to the honorable Vice Chancellor of BOU, Professor Dr. M. Farid Ahmed who gave us the opportunity to write the book. We are also grateful to the former Dean, School of Business, BOU Professor Dr. M. Ekramul Hoque who gave us the most needed support and enthusiasm to write this book. S. M. Miraj Ahmmod, Associate Professor, School of Business, BOU has made us indebted by his untiring efforts in editing and style editing each and every lesson diligently and meticulously. Our thanks are also due to Mohammad Wahiduzzaman Howlader, Word Processing Operator of the School of Business, for doing very best to complete the task of desktop processing on time.

We shall feel rewarded for our labor if both general readers and self-learners find this book worthwhile and useful.

Professor Dr. M. A. Taher
Dr. Mohammed Shamim Uddin Khan

CONTENTS

Theory of Set

1

This unit aims at explaining the set theory. Under the set theory, the topics covered are nature of set, types of sets, Venn diagram, basic set operations. Ample examples have been given in the lessons to demonstrate the application of set theory in practical contexts.

Blank Page

Lesson-1: Meaning, Methods and Types of Set

After studying this lesson, you should be able to:

- Identify sets and its elements;
- Apply the methods of describing a sets;
- Define and explain different types of sets.

Introduction

Mathematics speaks in the language of sets because it lies at the foundations of mathematics. Set is an undefined term, just as point and line are undefined in geometry.

Meaning of Sets and Element

A set is a collection definite and well distinguished objec

A set is understood to be a collection of objects. In other way, a set is a collection of definite and well distinguished objects. Each object belonging to a set is known as an element of the set.

Generally capital letters *A, B, C, X, Y*....etc. are used to denote a set and small letters *a, b, c, x, y,* ...etc are used to denote elements of a set.

A set may be described by listing its members/elements between the symbols {and}, which are called set braces. Thus, the expression {1, 2, 3, 4} is read as: The set of 1, 2, 3 and 4. The elements of the set are 1, 2, 3 and 4. The symbol for set elements is $\in$. Thus $1 \in \{1, 2, 3, 4\}$ is read as: 1 is an element of {1, 2, 3, 4}. The symbol $\notin$ is the negation of $\in$. Thus $6 \notin \{1, 2, 3, 4\}$ is read as: 6 is not an element of {1, 2, 3, 4}.

Methods of Describing a Set

A set can be described in the following two ways:

All the elements of set are enclosed by braces.

(1) Tabular Method: In this method, all the elements of the set are enclosed by set braces. For example,

(a) A set of vowels; A = {a, e, i, o, u}

(b) A set of even numbers; A = {2, 4, 6, ...}

(c) A set of first five letters of alphabet; A = {a, b, c, d, e}

(d) A set of odd numbers between 10 and 20; A = {11, 13, 15, 17, 19}

Elements of the set can be described o the basis of specifi characteristics of t

(2) Selector / Set-builder Notation Method: In this method, elements of the set can be described on the basis of specific characteristics of the elements. For example, let if x is the element of a set, then the above four sets can be expressed in the following way:

(a) A = {x | x is a vowel of English alphabet}

(b) A = {x | x is an even number}

(c) A = {x | x is a letter of the first five alphabet in English}

(d) A = {x | x is an odd number between 10 and 20}

In this case, the vertical line "|" after x is to be read as "such that".

Types of Sets

A set can be classified on the basis of special features of elements. There are different types of sets which are discussed below:

(i) Null, Empty or Void Set: A set having no element is known as null, empty or void set. It is denoted by Ø. For example,

(i) A = {x | x is an odd integers divisible by 2}

(ii) A = {x | x^2 = 4, x is odd}

A is the empty set in the above two cases.

t is finite if it sists of a specific ber of different nents.

(ii) Finite Set: A set is finite if it consists of a specific number of different elements, i.e. the counting process of the different members/elements of the set can come to an end. For examples,

(i) A = {1, 2, 3, 4, 5} (ii) A = {a, e, i, o, u}

then the sets are finite, because the elements can be counted by a finite number.

(iii) Infinite Set: If the elements of a set cannot be counted in a finite number, the set is called an infinite set. For example,

(a) Let A = {1, 2, 3, 4 ......}

(b) Let A = {x | x is a positive integer divisible by 5},

then the sets are infinite, as the process of counting the elements of these sets would be endless.

ery element in a A is also the nent of a set B, A is called a

(iv) Sub Sets: If every element in a set A is also the element of a set B, then A is called a subset of B. We denote the relationship by writing A ⊆ B, which can also be read as “A is contained in B.” For example,

A = {1, 2, 3, 4......}

B = {x | x is a positive even number}

C = {x | x is a positive odd number}

In this case B ⊆ A and C ⊆ A, because all the positive even and odd numbers are included in the set A.

(v) Proper subset: Since every set A is a subset of itself, we call B is a proper subset of A if B is a subset of A and B is not equal to A. If B is a proper subset of A, it can be represented symbolically as B ⊂ A. For example,

A = {a, b, c, d}, B = {a, c, b, d, c, a}

C = {a, c, d, a, d, a}

In this case, C ⊂ A and C ⊂ B, because the elements of C set are included in the sets A and B, but the element ‘b’ in of A and B sets is not element of C set.

(vi) Equal sets: Two sets A and B are said to be equal if every element which belongs to A also belongs to B, and if every element which belongs to B, also belongs to A. We denote the equality of sets A and B by ‘A = B’. For example, let A = {2, 3, 4}, B = {4, 2, 3}, C = {2, 2. 3, 4}, then A = B = C, since each element which belongs to any one of the sets also belongs to the other two sets.

If the elements of one set can be put into one to one corresponde-nce with the elements of another set, then the two sets are called

(**vii) Equivalent sets:** If the elements of one set can be put into one to one correspondence with the elements of another set, then the two sets are called equivalent sets. For example,

Let A = {a, b, c, d, e, f} and B = {1, 2, 3, 4, 5, 6}

In this case, the elements of set A can be put into one to one correspondence with those of set B. Hence the two sets are equivalent. It is denoted by A ≡ B.

(viii)Unit set/singleton: A set containing only one element is called a unit set or singleton. For example,

(a) A = {a}

(b) B = {x | x is a number between 27 and 34 divisible by 10}

In B set, 30 is the only number between 27 and 34 which is divisible by 10.

(ix) Power set: The set of all the subsets of a given set A is called the power set of A. We denote the power set of A by P(A). The power set is denoted by the fact that 'if A has n elements then its power set P(A) contains exactly 2^n elements'.

For example, let A = {a, b, c} then its subset are {a}, {b}, {c}, {a,b}, {b,c}, {c,a} {a,b,c}, {Ø}

∴ P (A) = [{a}, {b}, {c}, {a,b), {b,c}, {c,a}, {a,b,c}, {Ø}]

(x) Disjoint sets: If the sets A and B have no element in common, i.e., if no element of A is in B and no element of B is in A, then we say that A and B are disjoint.

For example, let A = {3, 4, 5} and B= {8, 9, 10, 11}, then A and B sets are disjoint because there is no element common in these two sets.

(xi) Universal sets: Usually, only certain objects are under discussion at one time. The universal set is the set of all objects under discussion. It is denoted by U or I. For example, in human population studies, the universal set consists of all the people in the world.

The universal set is the set of all objects under discussion.

Questions for Review

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. How would you define set? Identify some of the characteristics of sets. Is there any distinction between set and element?
2. Define the following with examples:

   Null set, finite set, infinite set, disjoint set, equal sets, equivalent sets, venn diagram, universal set.
3. (a) What is a subset and proper subset.

   (b) Find the power set of A = {1, 2, 3, 4}
4. List the elements of the following sets:

   (a) The set of all integers whose squares are less than 30;

   (b) The set of integers satisfying the equation $x^2 - 7x + 10 = 0$;

   (c) The set of all positive integers which are divisible by 5 and smaller than 78.
5. State whether each of the following sets is finite or infinite. When the set is finite indicate the number of elements it possesses;

   (a) The set of odd positive integers;

   (b) The set of all integers, whose squares are less than 45,

   (c) The set of integers satisfying the equation $x^2 - 5x + 6 = 0$

   (d) The set of students in your class who are taller than 7 feet.

Multiple Choice Questions (√ the appropriate answer)

1. A set is:
   (a) a collection of objects
   (b) a group of objects
   (c) a well defined collection of objects.

2. If A = {2}, which of the following statements is correct?
   (a) A = 2 (b) $2 \in A$ (c) $\{2\} \in A$.

3. The total number of elements in the power set of a set A containing 7 elements is:
   (a) 64 (b) 49 (c) 128

4. The number of all possible proper subsets of {2, 3, 5} is
   (a) 3 (b) 7 (c) 8.

5. Which one of the following is a finite set?
   (a) $\{x : x = y^2, y > 3\}$
   (b) $\{x : x = 2y, 30 < y < 40\}$
   (c) $\{x : x = y^3\}$

6. Which of the following pairs of sets is disjoint?

(a) {0, 1, 2} and {0, –1, –2}
(b) {1, 3, 4, 5} and {3, 5, 7}
(c) {1, 2, 3} and {–1, –2, –3}

Lesson-2: Venn Diagrams

After studying this lesson, you should be able to:

- Draw a Venn diagram of any set;
- Explain the nature of Venn diagram;
- Apply laws of sets for set operations;
- Explain the relationship between sets by using Venn diagram.

Venn Diagrams

Generally Venn diagram is used to help visualize any set and the relationship between sets. It is usually bounded by a circle. With the help of Venn diagram we can easily illustrate various set operations.

Following is the Venn diagram (Fig.1) of three sets A, B and C:

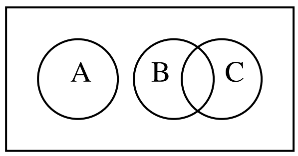


Fig.1

Laws of Algebra of Sets

Basic set operations viz. union, intersection and complement satisfy some laws, known as Laws of Algebra of Sets. We state below these laws of algebra of sets:

Basic set operations viz. union, intersec- tion and complement satisfy some laws, known as Laws of

1. **Idempotent Laws:** For any set A, we have (i) $A \cup A = A$, (ii) $A \cap A = A$.
2. **Commutative Laws:** For any two sets A and B, we have (i) $A \cup B = B \cup A$, (ii) $A \cap B = B \cap A$.
3. **Associative Laws:** For any three sets A, B and C, we have, (i) $A \cup (B \cup C) = (A \cup B) \cup C$, (ii) $A \cap (B \cap C) = (A \cap B) \cap C$.
4. **Distributive Laws:** (i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, (ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
5. **De Morgan's Laws:** For any two sets A and B, we have (i) $(A \cup B)^c = A^c \cap B^c$, (ii) $(A \cap B)^c = (A^c \cup B^c)$
6. **Identity Laws:** Let U be the universal set, ϕ be the null set and A be any subset of U. Then, (i) $A \cup U = U$, (ii) $A \cap U = A$, (iii) $A \cup \phi = A$, (iv) $A \cap \phi = \phi$.
7. **Complement Law :** with the same notation given in (6) above, we have (i) $A \cup A^c = U$, (ii) $(A \cup U)^c = \phi$, (iii) $(A^c)^c = A$, (iv) $U^c = \phi$, (v) $\phi^c = U$, where A^c is the complement of A.

Note: We observe similarity is some laws of the set theory with the ordinary algebraic laws of real numbers. If *a, b, c* are real numbers, we have following laws of algebra of numbers:

(i) $a + b = b + a$,
(ii) $a \times b = b \times a$,
(iii) $a + (b + c) = (a + b) + c$

Addition (+) and multiplication (×) notations of algebra of real numbers are replaced respectively by union (∪) and intersection (∩).

(iv) $a \times (b \times c) = (a \times b) \times c$

(v) $a \times (b + c) = a \times b + a \times c$.

If addition (+) and multiplication (×) notations of algebra of real numbers are replaced respectively by union (∪) and intersection (∩) notations of the set theory and the real numbers *a, b, c* are also replaced by the sets A, B, and C respectively, we obtain the following laws of algebra of sets:

(i) $A \cup B = B \cup A$

(ii) $A \cap B = B \cap A$

(iii) $A \cup (B \cup C) = (A \cup B) \cup C$

(iv) $A \cap (B \cap C) = (A \cap B) \cap C$

(v) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Some laws of algebra of sets differ from algebra of real numbers.

But some laws of algebra of sets differ from algebra of real numbers. For example, in ordinary algebra of real numbers, we have, (i) $a + a = 2a$, (ii) $a \times a = a^2$. But in algebra of sets, we have, (i) $A \cup A = A$, (ii) $A \cap A = A$.

In algebra of numbers, addition does not distribute across multiplication, i.e., for three real numbers *a, b* and *c*, $[a + (b \times c)] \neq (a + b) \times (a + c)$.

But in algebra of sets, union distributes across intersection, i.e., for three sets A, B and C we have $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Example-1:

Using Venn diagram, verify that $A \cap (B \cap C) = (A \cap B) \cap C$

Solution:

LHS: Assume that the rectangular regions in Figs.-2, 3, 4 and 5 represent the universal set U and its subsets A, B and C in each diagram are represented by circular regions.

In Fig.-2, the set A has been shaded by horizontal straight lines and the set $(B \cap C)$ has been shaded by vertical straight lines (i.e., the region common to both the sets B and C). Then by definition, the cross hatched region (i.e., the region where the horizontal and vertical lines intersect) represents the set $A \cap (B \cap C)$. The region representing this set has been shaded separately by slanting lines in Fig.-3.

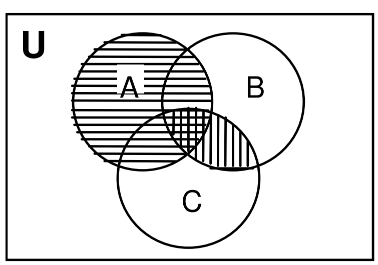


Fig. 2

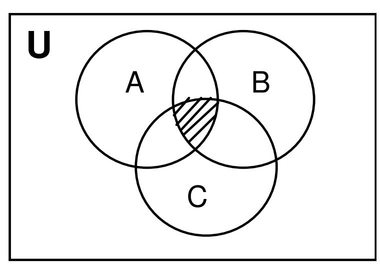


Fig. 3

RHS: In Fig.-4, the set $(A \cap B)$ has been shaded by horizontal lines (i.e., the region common to both the sets A and B) and the set C has been shaded by vertical straight lines. Then by definition, the cross hatched region (i.e., the region where the horizontal and vertical lines intersect) represents the set (A ∩ B) ∩ C. The region representing this set has been shaded separately by slanting lines in Fig.-5.

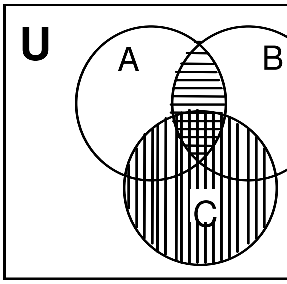


Fig.4

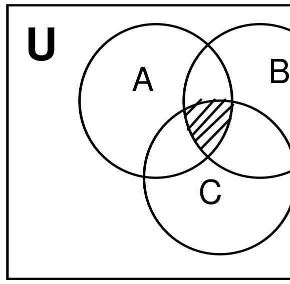


Fig.5

From Figs.-3 and 5, we see that the regions representing the sets [A ∩ (B ∩ C)] and [(A ∩ B) ∩ C] are identical. This verifies that A ∩ (B ∩ C) = (A ∩ B) ∩ C.

Example-2:

Using Venn diagram, verify that A ∪ (B ∩ C) = (A ∪ B) ∩ (A ∪ C)

Solution:

LHS: Assume that the rectangular regions in Figs.-6, 7, 8 and 9 represent the universal set U and its subsets A, B and C in each diagram are represented by circular regions.

In Fig.-6, the set A has been shaded by cross of horizontal and vertical lines. Set C has been shaded by horizontal straight lines and the set B has been shaded by vertical straight lines (i.e., the region common to both the sets B and C becomes a cross hatched region). Then by definition, the total cross hatched region represents the set A ∪ (B ∩ C).

The region representing this set has been shaded separately by slanting straight lines in Fig.-7.

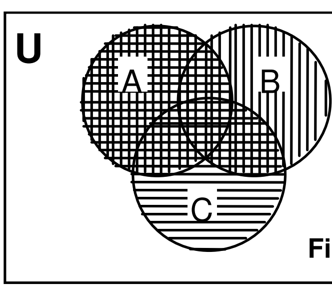


Fig. 6

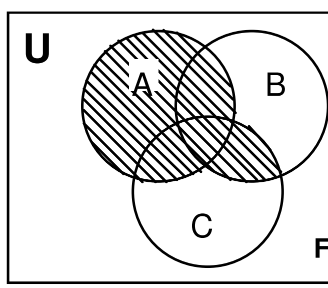


Fig. 7

RHS: In Fig.-8, the set (A ∪ B) has been shaded by vertical straight lines (i.e., the total region enclosed by the sets A and B) and the set (A ∪ C) has been shaded by horizontal straight lines (i.e., the total region enclosed by the sets A and C). Then by definition, the cross hatched region (i.e., the region where the horizontal and vertical lines intersect) represents the set (A ∪ B) ∩ (A ∪ C).The region representing this set has been shaded separately by slanting lines in Fig.-7.

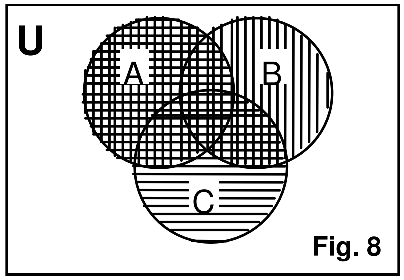


Fig. 8

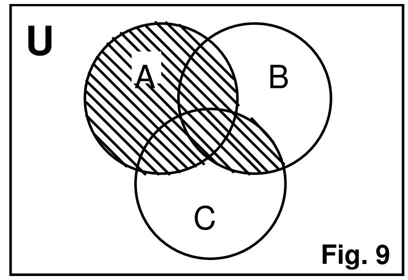


Fig. 9

From Figs.-7 and 9, we see that the regions representing the sets $A \cup (B \cap C)$ and $(A \cup B) \cap (A \cup C)$ are identical. This verifies that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

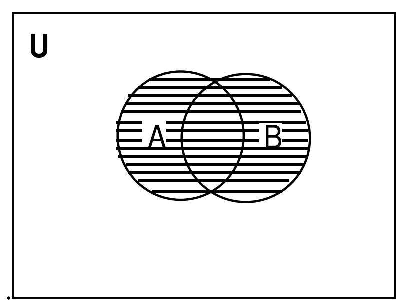
Example-3: [De Morgan's Laws]

Using Venn diagrams, verify that $(A \cap B)^c = A^c \cap B^c$

Solution:

LHS: Assume that the rectangular regions in Figs.-10, 11, 12 and 13 represent the universal set U and its subsets A and B in each diagram are represented by circular regions.

In Fig.-10, the set $A \cup B$ has been shaded by horizontal straight lines (i.e., the total region enclosed by the sets A and B). Then by definition, the region of the rectangle outside the shaded region represents the set $(A \cap B)^c$ (i.e. the complement of $A \cup B$). The region represented by $(A \cap B)^c$ has been shaded separately by slanting lines in Fig.-11.



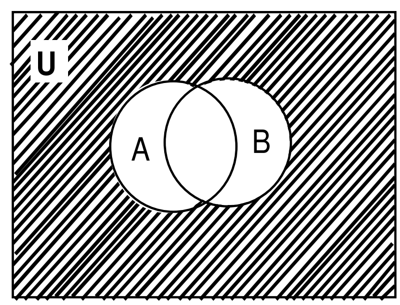


Fig.10 **Fig.11**

RHS: In Fig.-12, the set A^c has been shaded by horizontal straight lines (i.e., the region of the rectangle outside the set A) and the set B^c has been shaded by vertical straight lines (i.e., the region of the rectangle outside the set B). Then by definition, the cross hatched region (i.e., the region where the horizontal and vertical lines intersect) represents the set $A^c \cap B^c$. The region represented by the set $A^c \cap B^c$ has been shaded separately by slanting lines in Fig.-13.

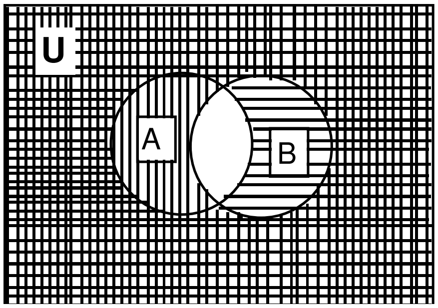


Fig.12

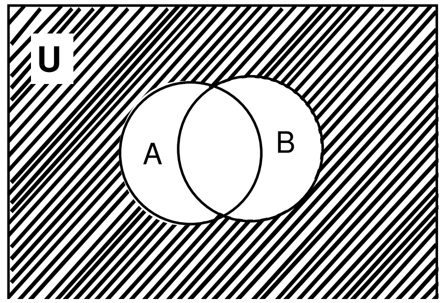


Fig.13

From Figs.-11 and 13, we see that the region representing the sets $(A \cup B)^c$ and $A^c \cap B^c$ are identical. This verifies that $(A \cup B)^c = A^c \cap B^c$

Questions for Review

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. What are the laws of algebra in set theory?

2. Using Venn diagram verify that $A \cup (B \cup C) = (A \cup B) \cup C$

3. Prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ using Venn diagram.

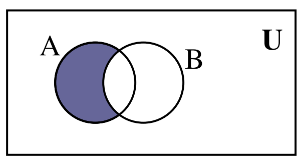
4. Prove that $(A \cap B)^C = A^C \cup B^C$ using Venn diagram.

5. Using Venn diagram show that $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$

Multiple Choice Questions (√ the appropriate answer)

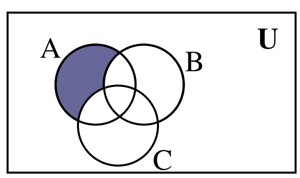
1. The shaded region in the adjoining diagram is:

(a) $A - B$ (b) $B - A$ (c) A^C



2. The shaded region in the adjoining diagram is:

(a) $A \cap (B \cup C)$ (b) $A - (B \cup C)$ (c) $A \cap (B - C)$



Lesson-3: Addition, Subtraction and Complement of Sets

After studying this lesson, you should be able to:

- Apply the addition operation of sets;
- Apply the subtraction operation of sets;
- Apply the complement operation of sets.

Introduction

Basic set operations will help the mathematician in identifying common elements or uncommon elements or differences of elements between two or more sets. It has been discussed as under:

Union of Sets

union of sets X Y is the set of all nents, which ng to X or to Y or

The union of sets X and Y is the set of all elements, which belong to X or to Y or to both. We denote the union of X and Y by $(X \cup Y)$, which is read as 'X union Y'. The union of X and Y may also be defined concisely by, $X \cup Y = \{x : x \in X \text{ or } Y\}$

Example-1:

Let X = {1, 2, 3, 4, 5, 6} and Y = { 3, 4, 5, 6, 7, 8}

then $X \cup Y = \{1, 2, 3, 4, 5, 6, 7, 8\}$ (According to the tabular method)
$X \cup Y = \{a : a \in I \text{ and } 1 \leq a \leq 8\}$ (According to selector method).

Properties

The important properties of the union of two or more sets are:

- The individual sets composing a union are elements/ members of the union, In other words $X \subseteq (X \cup Y)$ and $Y \subseteq (X \cup Y)$
- It has an identity property in an empty/null set. $\therefore$ $X \cup Ø = X$, for every set X.
- Union of a set with itself is the set itself, i.e., $X \cup Y = X$, for every set X.
- It has a commutative property, i.e., for any two sets X and Y, $X \cup Y = Y \cup X$
- It has an associative property, i.e., for any three sets X, Y and Z,

  $(X \cup Y) \cup Z = X \cup (Y \cup Z)$
- If $Y \subseteq X$, then $X \cup Y = X$ and if $x \subseteq y$, then $X \cup Y = Y$.
- $X \cup Y = Ø$, then $X = Ø$ and $Y = Ø$, in other words, both are null sets.
- $X \cap Y$ is the proper subset of X and X is the proper subset of $X \cup Y$. i.e., $(X \cap Y) \subset X \subset (X \cup Y)$.

Intersection of Sets:

intersection of X and Y is the set lements, which are mon to X and Y.

The intersection of sets X and Y is the set of elements, which are common to X and Y, that is, those elements which belong to X and which also belong to Y. We denote the intersection of X and Y by X ∩ Y, which is read as 'X intersection Y'.

The intersection of X and Y may also be defined concisely by X ∩ Y = {b : b ∈ X, b ∈ y}

Example-2:

Let X = {2, 3, 4, 5, 6, 7) and Y = {3, 4, 5, 6, 7, 8, 9}
Then X ∩ Y = {3, 4, 5, 6, 7} (According to tabular method)

X ∩ Y = {b : b ∈ I and 3 ≤ b ≤ 7 } (According to selector method)

Properties

The important characteristics of intersection of sets are as follows:

- X ∩ Y is the subset of both the set X and the set Y,

  i.e. (X ∩ Y) ⊆ X and (X ∩ Y) ⊆ Y.
- Intersection of any set with an empty set is the null set, i.e., X ∩ Ø = Ø for every set X.
- Intersection of a set with itself is the set itself, i.e. (X ∩ Y) = X, for every set X.
- Intersection has commutative property, i.e., X ∩ Y = Y ∩ X.
- Intersection has associative property. For any three sets X, Y and Z,

  (X ∩ Y) ∩ Z = X ∩ (Y ∩ Z)
- If X ⊆ Y, then X ∩ Y = X and Y ⊆ X, than X ∩ Y = Y. For example, if X = {2, 3} and Y = {2, 3, 4, 5, 6}, then X is the subset of Y, i.e., X ⊆ Y. In this case X ∩ Y = {2, 3}, because 2 and 3 are the common elements of X and Y sets. Therefore, X ∩ Y = X.
- If X ⊆ Y and Y ⊆ X then X ⊆ (Y ∩ Z); because Y ⊆ Z then Y ∩ Z = Y :

Distributive Laws of Unions and Intersections of Sets

The distributive laws of unions and intersections of the sets can be illustrated as under:

(i) The laws of the algebra of sets mentioned that the union distributes over intersection which is not possible in ordinary algebra,

i.e., X ∪ (Y ∪ Z) = (X ∪ Y) ∩ (X ∪ Z)

Let X = {a, b, c, d, e}; Y = {c, d, e, f, g} and Z = {e, f, g, h, i}

then (Y ∩ Z) = {e, f, g} and X ∪ (Y ∩ Z) = {a, b, c, d, e, f, g}

On the other side, X ∪ Y = {a, b, c, d, e, f, g}; X ∪ Z = {a, b, c, d, e, f, g, h, i}

then (X ∪ Y) ∩ (X ∪ Z) ={a, b, c, d, e, f, g}

So, X ∪ (Y ∩ Z) = (X ∪ Y) ∩ (X ∪ Z)

(ii) The algebra of sets can be expressed that the intersection distribute over the union which is also there in ordinary algebra. i.e., X ∩ (Y ∪ Z) = (X ∩ Y) ∪ (X ∩ Z).

Let X = {1. 2, 3, 4}, Y = {2, 3, 4, 5} and Z = {3, 4, 5, 6}

Then $(Y \cap Z) = \{2, 3, 4, 5, 6\}$ and

$X \cap (Y \cup Z) = \{2, 3, 4\}$

On the other hand,

$(X \cup Y) = \{2, 3, 4\}$; $(X \cap Z) = \{3, 4\}$

$\therefore (X \cap Y) \cup (X \cap Z) = \{2, 3, 4\}$

So, $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$

Complement of a Set

The complement of a set X is the set of elements which do not belong to X, that is, the difference of the universal set U and X. We denote the complement of X by X^C or X'. The complement of X may also be defined concisely by, $X^C = U - X = \{x : x \in U, \ x \notin X\}$.

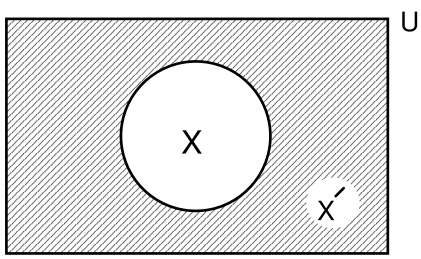
The complement of a set X is the set of elements which do not belong to X

Example-3:

Let $U = \{1, 2, 3, 4, 5, 6, 7\}$ and $X = \{2, 3, 4, 5\}$

Then $X^C = U - X = \{1, 6, 7\}$

The following Venn diagram showing the complement of a set:



The shaded region is X^C or $X' = (U - X)$

Properties

The important properties of complement of a set are:

- The intersection of a set X and its complement X′ is a null set, i.e., $X \cap X' = \emptyset$.
- The union of a set X and its complement X′ is the universal set, i.e., $X \cup X' = U$.
- The complement of the universal set is the empty set and the complement of the empty set is the universal set. Symbolically, $U' = \emptyset$ and $\emptyset' = U$.
- The complement of the complement of a set is the set itself. Symbolically, $(X')' = X$.
- If X is the proper subset of Y, then the complement of Y set is the proper subset of complement of X set. Symbolically, if $X \subset Y$, then $Y' \subset X'$.

- Expansion or contraction of sets is possible by taking into account the complements of a set. For example, $(X \cap Y) \cup (X \cap Y') = X$, and $(X \cup Y) \cap (X \cup Y') = X$.

Example-4:

Let $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6, 8\}$ and $C = \{3, 4, 5, 6\}$. Find (i) $A \cup B$, (ii) $A \cup C$, (iii) $B \cup C$, (iv) $B \cup B$, (v) $(A \cup B) \cup C$, (vi) $A \cup (B \cup C)$.

Solution:

(i) $A \cup B = \{1, 2, 3, 4, 6, 8\}$

(ii) $A \cup C = \{1, 2, 3, 4, 5, 6\}$

(iii) $B \cup C = \{2, 3, 4, 5, 6, 8\}$

(iv) $B \cup B = \{2, 4, 6, 8\}$

(v) $(A \cup B) \cup C = \{1, 2, 3, 4, 5, 6, 8\}$

(vi) $A \cup (B \cup C) = \{1, 2, 3, 4, 5, 6, 8\}$

Example-5:

Let $A = \{2, 3, 4, 5\}$, $B = \{3, 5, 7, 8\}$ and $C = \{4, 5, 6, 7, 8\}$

Find (i) $A \cap B$, (ii) $A \cap C$, (iii) $B \cap C$, (iv) $B \cap B$, (v) $(A \cap B) \cap C$, (vi) $A \cap (B \cap C)$.

Solution:

(i) $A \cap B = \{3, 5\}$

(ii) $A \cap C = \{4, 5\}$

(iii) $B \cap C = \{5, 7, 8\}$.

(iv) $B \cap B = \{3, 5, 7, 8\}$

(v) $(A \cap B) \cap C = \{5\}$

(vi) $A \cap (B \cap C) = \{5\}$.

Example-6:

Let $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6, 8\}$ and $C = \{3, 4, 5, 6\}$. Find (i) $A - B$; (ii) $C - A$; (iii) $B - C$; (iv) $B - A$; (v) $B - B$.

Solution:

(i) $A - B = \{1, 3\}$

(ii) $C - A = \{5, 6\}$

(iii) $B - C = \{2, 8\}$

(iv) B – A = {6, 8}

(v) B – B = {ϕ}.

Questions for Review

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. Define the following with examples:

   (a) Union of sets, (b) Intersection of sets, and (c) Complement of a set.

2. Let the universal set and sets A, B and C are as follows:

   U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

   A = {1, 2, 3, 5}

   B = {2, 5, 6, 8}

   C = {5, 6, 8, 9, 10}

   Find (i) $A \cup B \cup C$; (ii) $(A \cup B \cup C)'$; (iii) $A \cap B \cap C$; (iv) C'; (v) B'; (vi) A'.

3. Let U = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}

   A = {1, 2, 5, 7}, B = {0, 1, 4, 6}, C = {3, 4, 5, 7}

   Show that $(A \cup B \cup C)' = A' \cap B' \cap C'$

4. If the universal set U = $\{x : x \in N \text{ and } 1 \le x \le 10\}$

   A = $\{x : x \in N \text{ and } 1 \le x \le 8\}$

   B = {x : x is a natural number, which is less than 10 and divisible by 3}

   C = {1, 2, 3, 5, 6}

   Find (i) A'; (ii) $A \cup B$; (iii) $A \cap C$; (iv) $(A \cup C)'$; (v) $B' \cap C$.

Multiple Choice Questions (√ the appropriate answer)

1. If A = {1, 2, 3, 4} and B = {5, 6}, then $A \cup B$ is

   (a) {1, 2, 3, 4, 5, 6} (b) ϕ, (c) {1, 2, 3, 4, 6}

2. Let A = {0, 1, 3, 4}, B = {5, 6, 1, 3, 9} and C = {0, 1, 2, 3, 9, 13}. Then, $(A \cap B) \cup C$ is :

   (a) {0, 1, 3} (b) 0, 1, 2, 3, 9, 13} (c) {1, 3}

3. Let A = {1, 2, 3, 4, 5}, B = {2, 4, 6, 8} and C = {3, 4, 5, 6} then, $(A \cup B) \cap C$ is:

   (a) {1, 2, 3, 4, 5, 6, 8} (b) {3, 4, 5, 6} (c) {3, 4}

4. Which of the following is a true statement?

   (a) $(A \cup B)' = (A' \cup B')$

   (b) $(A \cup B)' = A' \cap B'$

   (c) $(A \cup B)' = A - B'$

5. If U = {1, 2, 3, 4, 5, 6}; A = {3, 5} , then A' is equal to :

   (a) {1, 2, 4, 6} (b) {3, 5, 6} (c) {1, 2, 3}

6. Let U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10} be the universal set and A = {2, 4, 6}, B = {1, 3, 7). Then $A' \cap B'$ is equal to:
   (a) {2, 4, 5, 6, 8, 9, 10}
   (b) {5, 8, 9, 10}
   (c) {1, 3, 5, 7, 8, 9, 10}

Lesson-4: Difference and Product of Sets

After studying this lesson, you should be able to:

- State the difference of sets;
- State the product of sets;
- Explain the presentation of sets with corresponding set notation.

Difference of Two Sets

The difference of set Y from set X is the set of elements, which belong to X but which do not belong to Y. We denote the difference of X and Y by (X ~ Y), which is read as: X difference Y, or simply, 'X minus Y'. The difference of X and Y may also be defined concisely by, $X \sim Y = \{a : a \in X, a \notin Y\}$.

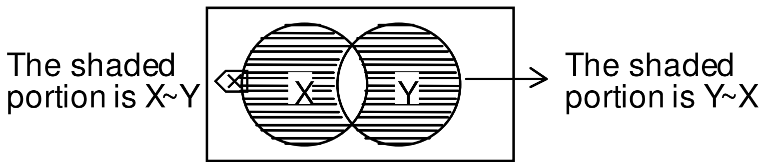
The difference of se from set X is the se elements, which belong to X but wh do not belong to Y.

For example**:** Let X = {a, b, c, d, e, f} and Y = {d, e, f, g, h}

Then X ~ Y = {a, b, c}

and Y ~ X = {g, h}

The difference of two set can be shown by Venn diagram as under:



Properties

The important properties of the difference of two sets are as under:

- $X - Y$ is the subset of X, i.e., $(X - Y) \subseteq X$ and $(Y - X)$ is the subset of Y, i.e., $(Y - X) \subseteq Y$.
- $(X - Y)$, $(X \cap Y)$ and $(Y - X)$ are mutually disjoints.
- $X - (X - Y) = (X \cap Y)$ and $Y - (Y - X) = X \cap Y$.

Product of Two Sets

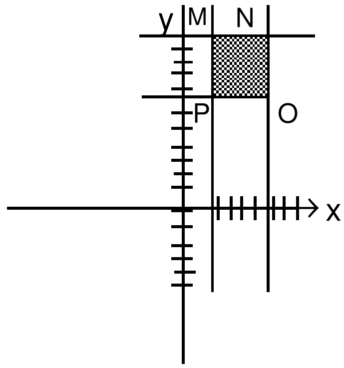
Let X and Y be two sets. The product of sets X and Y consists of all ordered pairs where $\{(x, y): x \in X \text{ and } y \in Y\}$. It is denoted by $(X \times Y)$, which is read as "X cross Y". The product of sets X and Y may also be defined concisely by, $(X \times Y) = \{(x, y) : x \in X, y \in Y\}$.

The product of sets and Y consists of a ordered pairs wher {(x, y): x ∈ X and y∈Y}.

The product of sets $(X \times Y)$ is also called Cartesian product of X and Y.

For example: Let X = {1, 2, 3} and Y = {5, 6, 7}

Then X.Y = {(1,5), (1,6), (1,7), (2,5) (2,6) (2,7), (3,5), (3,6) (3,7)}. The Cartesian product of X and Y sets can be displayed in the following rectangular co-ordinate system.



The shaded portion is XY and MNOP is the required rectangular system of XY.

Properties

The important properties of a Cartesian product are as follows:

- X.Y and Y.X have the same number of elements but X.Y ≠ Y.X, unless X = Y. Thus the Cartesian product of two sets is commutative if the two sets are equal.
- In the product of sets Y.X, the first component of ordered pairs are taken from Y and the second from X.
- If X and Y are disjoint sets, then X.Y and Y.X are also disjoint.
- If the set X consists of *m* elements $x_1, x_2, ...x_m$ and set Y consists of the *n* elements $y_1, y_2, y_3, ...y_n$, then the product sets X.Y consists of *mn* elements.
- If either X or Y is null then the set X.Y is also a null set.
- If either X or Y is infinite and the other is a non-empty set, then X.Y is also an infinite set.
- If $X \subset Y$, then $X.Z \subset Y.Z$
- If $X \subset Y$ and $Z \subset D$, then $X.Z \subset Y.D$.
- If $X \subseteq Y$ then $X.Y \Rightarrow (X.Y) \cap (Y.X)$
- If X, Y and Z be any three sets, then $X.(Y \cap Z) = (X.Y) \cap (X.Z)$
- If X, Y and Z be any three sets, then $X.(Y \cup Z) = (X.Y) \cup (X.Z)$
- $(X.Y) \cap (Z.D) = (X \cap Z) \times (Y \cap D)$.

The following examples contain some model applications of set theory.

Example-1:

Let A = {a, b}, B = {2, 3} and C = {3, 4}. Find (i) $A \times (B \cup C)$; (ii) $(A \times B) \cup (A \times C)$; (iii) $A \times (B \cap C)$; (iv) $(A \times B) \cap (A \times C)$.

Solution:

(i) $A \times (B \cup C) = (a, b) \times (2, 3, 4)$
$= \{(a,2), (a,3), (a,4), (b,2), (b,3), (b,4)\}$

(ii) $(A \times B) \cup (A \times C)$
$(A \times B) = \{(a, b) \times (2, 3)\} = \{(a,2), (a,3), (b,2), (b,3)\}$
$(A \times C) = \{(a, b) \times (3, 4)\} = \{(a,3)\ (a,4)(b,3)(b,4)\}$.
$(A \times B) \cup (A \times C) = \{(a,2), (a,3), (a,4), (b,2), (b,3), (b,4)\}$

(iii) $A \times (B \cap C) = \{(a, b) \times (3)\} = \{(a, 3), (b, 3)\}$

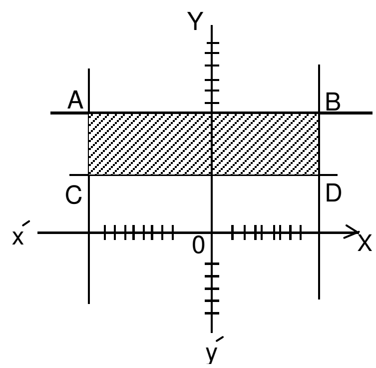
(iv) $(A \times B) \cap (A \times C) = \{(a, 3), (b, 3)\}$.

Example-2:

Let R represent the set of all rational numbers and
$X = \{x : x \in R \text{ and } -4 \leq x < 3.5\}$
$Y = \{y : y \in R \text{ and } 1.5 < y \leq 4.37\}$
(i) Express $X \cup Y$ and $X \cap Y$.
(ii) Draw a rectangular coordinate system and show XY on it.

Solution:

(i) $X \cup Y = \{m : m \in R \text{ and } -4 \leq m \leq 4.37\}$
$X \cap Y = \{m : m \in R \text{ and } 1.5 < m < 3.5\}$
(ii) $X.Y = \{(x,y) : x \in X, y \in Y, -4 \leq x < 3.5 \text{ and } 1.5 < y \leq 4.37\}$
The following rectangular of coordinates shows X.Y in set notation.



So, the ABCD is the required rectangular coordinate system of X.Y.

Example-3:

Let R represent the set of all real number and.

$X = \{x \mid x \in R \text{ and } -1 \leq x < 2\}$

$Y = \{y \mid y \in R \text{ and } 0 \leq y \leq 3\}$

(i) Draw a rectangular coordinate system and show X.Y on it.

(ii) Draw another rectangular coordinate system and show Y.X on it.

Solution:

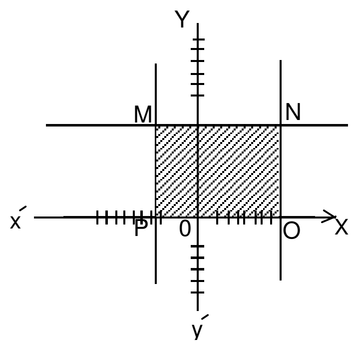
(i) Element of X set = $\{-1, 0, 1\}$
Element of Y set = $\{0, 1, 2, 3\}$
In set notation:

X.Y = {(-1,0), (-1,1), (-1,2), (-1,3), (0,0), (o,1), (0,2), (0,3), (1,0), (1,1), (1,2), (1,3)}

In expression:

X.Y = {$(x, y) \mid x \in$ X, $y \in$ Y, $-1 \leq x < 2$ and $0 \leq$ y 3}

The following rectangular of coordinates shows X.Y in set notation.



∴ MNOP is the required Rectangular system of X.Y.

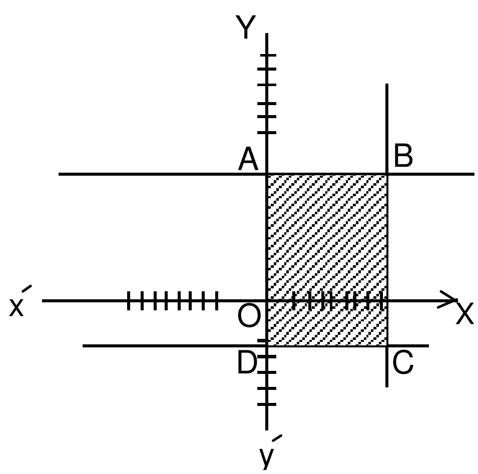
(ii) In set notation:

Y.X = {(0,1), (0,0), (0,1), (1,-1), (1,0), (1,1), (2,-1), (2,0), (2,1), (3,-1), (3,0), (3,1)}

In expression:

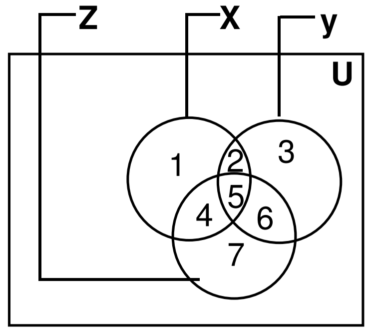
Y.X = {$(x, y) \mid x \in$ X, $y \in$ Y, $0 \leq$ y $\leq$ 3 and $-1 \leq x < 2$}

The following rectangular coordinate shows Y.X in expression.



So, ABCD is the required rectangular co-ordinate system of Y.X.

In the previous discussion we have learned about the union of sets, intersection of sets, complement of a set, difference of sets and product of two sets. Now the different parts of the three joint sets of X, Y and Z are expressed in mathematical way as under:



In the above Venn diagram,

(1) indicates $X \cap Y' \cap Z'$

(2) indicates $X \cap Y \cap Z'$

(3) indicates $X' \cap Y \cap Z'$

(4) indicates $X \cap Y' \cap Z$

(5) indicates $X \cap Y \cap Z$

(6) indicates $X' \cap Y \cap Z$

(7) indicates $X' \cap Y' \cap Z$

(8) indicates $X' \cap Y' \cap Z'$

Questions for Review

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. Define the following with examples:
   Product of two sets, Difference of two sets.
2. Let the universal set, U = {a, b, c, d, e, f, g}, X = {a, b, c, d, e}
   Y = {a, c, e, g} and Z = {b, e, f, g}
   Find (i) $X \cup Z$, (ii) $Y \cap X$, (iii) $Z \sim Y$, (iv) Y', (v) $X' - Y$, (vi) $Y' \cap Z$, (vii) $(X \sim Z)'$, (viii) $Z' \cap A$, (ix) $(X \sim Y)'$, (x) $(X \cap X')'$.
3. If M = {1, 2, 3}, N = {2, 3, 4}, O = {1, 3, 4} and P = {2, 4, 5}, Prove that $(M \times N)\ (O \times P) = (M \cap O) \times (N \cap P)$
4. Let R represents the set of all rational numbers and
   $X = \{x : x \in R \text{ and } -2 \leq x < 3.5\}$
   $Y = \{y : y \in R \text{ and } 1.5 < y \leq 4.32\}$
   (i) Express $X \cup Y$ and $X \cap Y$.
   (ii) Draw rectangular coordinate system and show (a) X.Y, and (b) Y.X on it.
5. Given A = {1, 3, 4, 7); B = {3, 7, 12}; C = {1, 5, 8}
   Write the following sets:
   (i) The set containing all elements that are members of A or members of B or members of both A & B.
   (ii) The set of elements that are members of both A and B.
   (iii) The set of elements that are members of both B and C.
   (iv) The set of elements that are members of A but not members of B.
   (v) The set of elements that are members of all three sets.

Multiple Choice Questions (√ the appropriate answer)

1. If A = {1, 2, 3, 4, 5, 6, 7, 8, 9}, B = {2, 4, 6, 7, 8} and C = (3, 4, 5, 8, 9, 10}, then $(A - B) \cup C$ is
   (a) {1, 3, 4, 5, 8, 9, 10}
   (b) {2, 4, 6, 7, 8}
   (c) {1, 3, 4, 5, 8, 9}
2. $A - (B \cup C)$ equals:
   (a) $(A - B) \cup (A - C)$
   (b) $(A \cap B) \cap (A - C)$
   (c) $(A - B) \cup C$.
3. If $U = \{x \in N : 1 \leq x \leq 10\}$, A = {1, 2, 3, 4}, B = {2, 3, 6, 10} then $(A - B)'$ is :
   (a) {1, 4}   (b) {2, 3}   (c) {2, 3, 5, 6, 7, 8, 9, 10}
4. $A \cap (B - C)$ is equal to:
   (a) $(A \cap B) - (A \cap C)$
   (b) $(A \cap B) - C$
   (c) $(A \cap B) - (A \cup C)$

5. If A = {2, 3, 5}, B = {4, 5, 6}, then $(A \cap B) \times A$ is
   (a) {2, 5}, (3, 5)}
   (b) {(5, 2), (5, 3), (5, 5)}
   (c) {(5, 2), (2, 5), (3, 5)}

6. If a set A has 5 elements and a set B has 10 elements, then the number of elements in $(A \times B)$ is:
   (a) 50 (b) 15 (c) 5.

Lesson-5: Applications of Set Theory to Solve Business Problems

After studying this lesson, you should be able to:

- Explain the relationship of sets;
- Apply the principles of set theory in business problems.

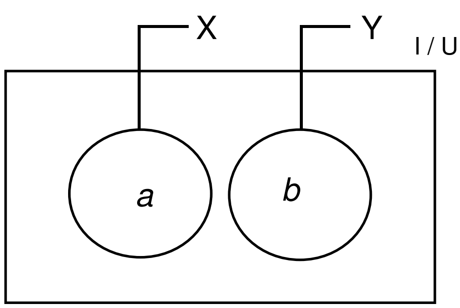
Introduction

The numbers of elements in a set X is denoted by 'n(X)'. Again, the number of elements in a set Y is expressed by n(Y). Here we derive a formula for $n(X \cup Y)$ in terms of n(X), n(Y) and $n(X \cap Y)$. First we observe that if X and Y set are disjoint, i.e., if $(X \cap Y) = \emptyset$, then $n(X \cup Y) = n(X) + n(Y)$

union of two joint are $n(X \cup Y) =$) $+ n(Y) - (X \cap Y)$.

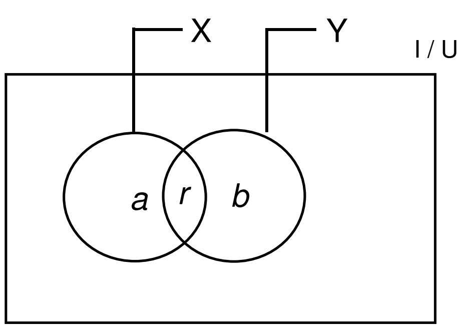
Later we take the case of the union of two finite sets which are not mutually disjoint, that is, there are some common elements between the two sets, i.e., the union of two joint sets are $n(X \cup Y) = n(X) + n(Y) - (X \cap Y)$.

The following Venn diagram presents two disjoint sets:



Here, the numbers of the elements of X set, n(X) = *a,* and Y set, n(Y) = *b*. If the sets are disjoint, then, $n(X \cup Y) = (a + b) = n(X) + n(Y)$

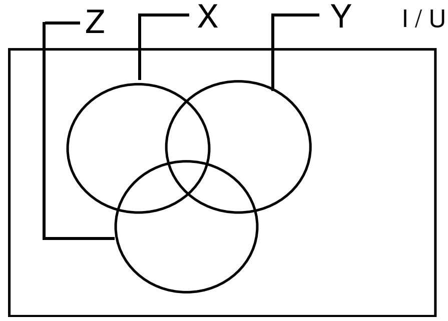
On the other hand, if 'r' is the common element of both the sets X and Y, the total elements of X set n(X) = a + r and Y set, n(Y) = b + r. i.e., $n(X \cup Y) = \{a, b, r\}$



But if $n(X \cup Y) = n(X) + n(Y)$, then the elements of $n(X \cup Y)$ are $\{(a + r) + (b + r)\} = \{a + b + r + r\}$. Here the element 'r ' will be deleted because it is added more than one time.

i.e., $n(X \cup Y) = [\{ a + r\} + \{ b + r\} - r] = n(X) + n(Y) - n(X \cap Y)$

Again for the union of any three sets X, Y and Z, which are mutually disjoint, we have, $n(X \cup Y \cup Z) = n(X) + n(Y) + n(Z)$. But when these three sets are joint, then the Venn diagram would be as under:



In the above diagram, the three sets are mutually joint. For the union of these three sets the elements are, $n(X \cup Y \cup Z)$.
Now, if we consider that X is a set and $(Y \cup Z)$ is another set, then as per union of sets, wet get:

$$n(X \cup Y \cup Z) = n\,[\{X \cup (Y \cup Z)\,].$$
$$= n(X) + n\,(Y \cup Z\,) - n[\,X \cap (Y \cup Z)\,]$$

Again $n(Y \cup Z) = n(Y) + n(Z) - n(Y \cap Z)$

and $n[X \cap (Y \cup Z\,) = n[(X \cap Y) \cup (X \cap Z)]$ (using distributive law)

$$= n(X \cap Y) + n(X \cap Z) - n[(X \cap Y) \cap (X \cap Z)]$$
$$= n(X \cap Y) + n(X \cap Z) - n(X \cap Y \cap Z)$$

Therefore

$$n(X \cup Y \cup Z) = n(X) + n(Y) + n(Z) - n\,(X \cap Y) - n(Y \cap Z) - n(X \cap Z) + n(X \cap Y \cap Z)$$

Here, $n(X) \Rightarrow$ The elements of X set

$n(Y) \Rightarrow$ The elements of Y set

$n(Z) \Rightarrow$ The elements of Z set

$n(X \cap Y) \Rightarrow$ The common elements of X and Y set.

$n(Y \cap Z) \Rightarrow$ The common elements of Y and Z set.

$n(X \cap Z) \Rightarrow$ The common elements of X and Z set.

$n(X \cap Y \cap Z) \Rightarrow$ The common elements of X, Y and Z set.

Now we have got an idea regarding the operation of set theory, which can be applied in the field of business.

The operation of se theory, which can b applied in the field business.

The following section of this lesson contains some model applications of set theory.

Example-1:

There are 1,500 students who appeared at the CMA examination under the ICMAB. Out of these students, 450 failed in Accounting, 500 failed in Business Mathematics and 475 failed in Costing. Those who failed in both Accounting and Business Mathematics were 300, those who failed in both Business Mathematics and Costing were 320 and those who failed in both Accounting and Costing were 350. The students who failed in all the three subjects were 250.

Find (i) How many students failed in at least any one of the subjects?

(ii) How many students failed in no subjects?

(iii) How many students failed in only one subjects?

(iv) How many students failed in both Accounting and Business Mathematics only?

Solution:

Let U is the set of the students who appeared at the CMA examination and A, B and C denote the set of students who failed in Accounting, Business Mathematics and Costing respectively. Now we are given,

$n(U) = 1500 \qquad n(A \cap B) = 300$

$n(A) = 450 \qquad n(B \cap C) = 320$

$n(B) = 500 \qquad n(A \cap C) = 350$

$n(C) = 475 \qquad n(A \cap B \cap C) = 250.$

(i) Number of students who failed in at least any one of the subjects.

Now, $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A\cap B) - n(B\cap C) - n(A\cap C) + n(A\cap B\cap C)$

$= 450 + 500 + 475 - 300 - 320 - 350 + 250$

$= (1675 - 970) = 705$

Therefore, the number of students who failed at least in any one of the subjects is 705.

(ii) Number of students who failed in no subjects.

$n(A \cap B \cap C)' = n(U) - n(A \cup B \cup C)$

$= (1500 - 705) = 795$

Hence the numbers of students who failed in no subjects is 795

(iii) Number of students who failed in only one subject:

Now, $n(A \cap B' \cap C') = n(A) - n(A \cap B) - n(A\cap C) + n(A \cap B \cap C)$

$= (450 - 300 - 350 + 250) = 50$

$n(A' \cap B \cap C') = n(B) - n(A \cap B) - n(B \cap C) + n(A \cap B \cap C)$

$= (500 - 300 - 320 + 250) = 130$

$n(A' \cap B' \cap C) = n(C) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$

$= (475 - 350 - 320 + 250) = (725 - 670) = 55$

Hence the number of students who failed in only one subject is,

$n(A \cap B' \cap C') + n(A' \cap B \cap C)' + (A' \cap B' \cap C)$

$= (50 + 130 + 55) = 235.$

(iv) Number of students who failed in both Accounting and Business mathematics only:

Now, $n(A \cap B \cap C') = n(A \cap B) - n(A \cap B \cap C)$

$= (300 - 250) = 50$

So the number of students who failed in both Accounting and Business Mathematics only is 50.

Example-2:

A Survey of 600 workers in a plant indicated that 410 owned their own houses, 500 owned cars, 550 owned televisions, 410 owned cars and televisions, 340 owned cars and houses, 370 owned houses and television and 300 owned all three. Illustrate by a Venn diagram and prove that the above data is not correct. What set is empty?

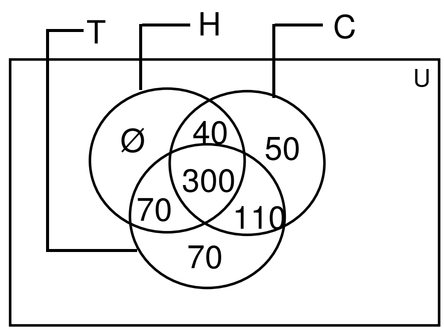
Solution:

Let U is the set of the workers who were surveyed and H, C and T are the sets of workers who owned their houses, cars and televisions respectively. Now we are given,

$n(U) = 600$; $n(H) = 410$; $n(C) = 500$; $n(T) = 550$; $n(C \cap T) = 410$;

$n(H \cap T) = 370$; $n(C \cap H) = 340$; $n(C \cap H \cap T) = 300$.

The following Venn diagram shows the result of the survey of ownership:



Hence, the total number of workers in the survey is,

$n(H \cup C \cup T) = n(H) + n(C) + n(T) - n(H \cap C) - n(C \cap T) - n(H \cap T) + n(H \cap C \cap T)$

$= (410 + 500 + 550 - 340 - 410 - 370 + 300) = 640$

This figure exceeds the total number of workers who were surveyed. Hence the given data is not correct or consistent.

In Venn diagram, $(H \cap C' \cap T')$ set is empty.

$n(H \cap C' \cap T') = n(H) - n(H \cap C) - n(H \cap T) + n(H \cap C \cap T)$

$= (410 - 340 - 370 + 300) = (710 - 710) = \emptyset.$

Example-3:

Mr. Arefin has 165 workers in process-X, 110 workers in process-Y and 97 workers in process-Z in his firm. Out of these workers, 281 workers are skilled in the activities of X and/or Y, 269 workers are skilled in the activities of Y and/or Z, 241 workers are skilled in the activities of X and/or Z. However, 44 workers are unskilled.

Accounts department of his firm has informed that the average monthly earnings of different types of workers are as follows:

Workers who are skilled in the activities of at least two processes: Tk.3,500

Workers who are skilled in the activities of any one process: Tk.2,500

Workers who are not skilled: Tk.1,500

Find the monthly amount of total earnings of all workers of the firm.

Solution:

Let U is the set of the workers who are skilled in all the processes. X, Y and Z are the sets of workers who are skilled in Process-X, Process-Y and Process–Z respectively.

We are given,

$n(X) = 165$; $n(Y) = 110$; $n(Z) = 97$; $n(X \cup Y) = 281$; $n(Y \cup Z) = 269$; $n(X \cup Z) = 241$.

The numbers of workers who are not skilled is 44.

Now, the total number of workers, $n(U) = (Px + Py + Pz) = (165+110+97) = 372$

The total number of the workers who are skilled in at least one of the processes;

$$n(X \cup Y \cup Z) = (372 - 44) = 328$$

The number of workers who are skilled in only X,

$$n(X \cup Y \cup Z) - n(Y \cup Z) = (328 - 269) = 59$$

The number of workers who are skilled in only Y,

$$n(X \cup Y \cup Z) - n(X \cup Z) = (328 - 241) = 87$$

The number of workers who are skilled in only Z,

$$n(X \cup Y \cup Z) - n(X \cup Y) = (328 - 281) = 47$$

So, the total number of workers who are skilled in only one process,

= (59 + 87 + 47) = 193.

Therefore the number of workers who are skilled in at least two processes is,

= (328 – 193) = 135.

Hence the monthly amount of total earnings of all workers of the firm is,

=135.(3,500) + 193.(2,500) + 44.(1,500)

= (4,72,500 + 482,500 + 66,000) = Tk.10,21,000.

Questions for Review

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. Prove that $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$.

2. In a survey, only 60% of 1000 questionnaires are found correct. Survey result indicates that only 42% prefer their present job responsibilities and 55% prefer their job environment. If 30% prefer both the job responsibility and job environment, how many do not prefer any one of these two?

3. In a survey of 100 families, the numbers that read the recent issues of various magazines were found to be as follows:

   Dhaka Courier 28; Readers Digest 30; Bangladesh Time 5; Courier and Readers Digest 8; Courier and Bangladesh Time 10. Readers Digest and Bangladesh Time 42; All the three magazines 3. With the help of set theory, find

   (i) How many read none of the three magazines?

   (ii) How many read Bangladesh Time as their only magazine?

   (iii) How many read Reader's Digest if and only if they read Bangladesh Time?

4. The production manager of MIC House, M. Nuruddin has 95 workers in Division-P, 80 workers in Division-Q and 120 workers in Division-R in his firm. Three different types of products are produced in these three divisions and workers in each division can easily perform the activities of that division. However, out of these workers, 25 workers can perform the activities of P and/or Q, 32 workers can perform the activities of Q and/or R, 39 workers can perform the activities of P and/or R. There are only 12 workers who can perform any activity of the three divisions. Due to change in the demand of the product of three divisions, M. Nuruddin has to shift workers from one division to another. In a 40-hour week, what would be the maximum labor hours in each division that can be worked by this work force?

Multiple Choice Questions (√ the appropriate answer)

1. In a group of persons, each one knows either Bengali or English. If 100 know Bengali, 50 know English and 30 know both, how many persons are there in the group?

   (a) 130, (b) 120, (c) 150

2. If 63% of Bangladeshi like milk and 76% like tea, how many like both?

   (a) 39%, (b) 26%, (c) 13%

3. In a group of 52 persons, 16 drink tea but not coffee and 33 drink tea. How many drink coffee but not tea?

   (a) 17, (b) 3, (c) 19

4. A dinner party is to be fixed for a group consisting of 100 persons. In this party, 50 persons do not prefer fish, 60 prefer chicken and 10 do not prefer either chicken or fish. The number of persons who prefer both fish and chicken is:

   (a) 30, (b) 10, (c) 20

5. In a class consisting of 100 students, 20 know English, 20 do not know Hindi and 10 know neither English nor Hindi. The number of students knowing both Hindi and English is:

   (a) 15, (b) 20, (c) 10

Logarithm

2

The purpose of this unit is to equip the learners with the concept of logarithm. Under the logarithm, the topics covered are nature of logarithm, laws of logarithm, change the base of logarithm, anti-logarithm and its operation followed by ample examples.

Blank Page

Lesson-1: Nature and Basic Laws of Logarithm

After studying this lesson, you should be able to:

- Discuss the nature of logarithm;
- Identify the basic laws of operation of logarithm;
- Explain the characteristics and mantissa of logarithm.

Meaning of a Logarithm

Logarithm is the important tool of modern mathematics. If $a^x = n$, then x is said to be the logarithm of the number *'n'* to the base *'a'*. Symbolically it can be expressed as follows: $log_a\ n = x$. In this case $a^x = n$ is an exponential form and $log_a\ n = x$ is a logarithmic form. The object of logarithm is to make common calculations less laborious and the method consists in replacing multiplication by addition and division by subtraction.

Logarithm to the base *'e'* is called *'natural logarithm'* and when the base is 10, the logarithm is called *'common logarithm'*. For example,

Logarithm to the base 'e' is called 'natural logarithm' and when the base is 10, the logarithm is called 'common logarithm'

(i) $5^3 = 125 \rightarrow \log_5 125 = 3$, i.e. the logarithm of 125 to the base 5 is equal to 3.

(ii) $(64)^{\frac{1}{6}} = 2 \rightarrow \log_2 64 = \frac{1}{6}$, i.e. the logarithm of 64 to the base 2 is equal to $\frac{1}{6}$.

Similarly,

Exponential form		Logarithmic form
$2^3 = 8$	$\rightarrow$	$log_2 8 = 3$
$10^2 = 100$	$\rightarrow$	$log_{10} 100 = 2$
$2^{-2} = \frac{1}{4}$	$\rightarrow$	$log_2 \frac{1}{4} = -2$
$3^0 = 1$	$\rightarrow$	$log_3 1 = 0$

or

Logarithmic form		Exponential form
$log_4 64 = 3$	$\rightarrow$	$4^3 = 64$
$log_p R = Q$	$\rightarrow$	$P^Q = R$
$log_{10} 10 = 1$	$\rightarrow$	$10^1 = 10$
$log_5 1 = 0$	$\rightarrow$	$5^0 = 1$

Fundamental Properties and Laws of Logarithms

The fundamental properties and laws of logarithm are as follows:

(1) The logarithm of the production of two factors is equal to the sum of their logarithms; i.e., $log_a mn = log_a m + log_a n.$

(2) The logarithm of quotient is equal to logarithm of the numerator minus the logarithm of the denominator; i.e., $log_a \left(\frac{m}{n}\right) = log_a m - log_a n.$

(3) The logarithm of any power of a number is equal to the product of the index of the power and the logarithm of the number; i.e. $log_a m^x = x \, log_a m.$

(4) Base changing formula: The formula which tells us how to change from one base to another is : $log_b n = \frac{\log_a n}{\log_a b}$

i.e., $(log_b n)(log_a b) = log_a n$

Characteristics and Mantissa of a Logarithm

The logarithm of a number consists of two parts. The first part is called characteristics and the second part are termed as mantissa

The logarithm of a number consists of two parts: (i) an integer positive, negative or zero (ii) a positive or negative proper fraction. The first part is called characteristics and the second part are termed as mantissa.

Since $10^0 = 1$ $\quad \therefore log\ 1 = 0$

$10^1 = 10$ $\quad \therefore log\ 10 = 1$

$10^2 = 100$ $\quad \therefore log\ 100 = 2$

$10^3 = 1000$ $\quad \therefore log\ 1000 = 3$

$10^4 = 10{,}000$ $\quad \therefore log\ 10{,}000 = 4$

Similarly, since $10^{-1} = \frac{1}{10} = 0.1,$ $\quad \therefore log\ 0.1 = -1$

$10^{-2} = \frac{1}{100} = 0.01,$ $\quad \therefore log\ 0.01 = -2$

$10^{-3} = \frac{1}{1000} = 0.001,$ $\quad \therefore log\ 0.001 = -3$

$10^{-4} = \frac{1}{10000} = 0.0001,$ $\quad \therefore log\ 0.0001 = -4.$

In general, the logarithm of a number containing *n* digits only in its integral part is $\{(n - 1) + a\}$ fraction and the logarithm of a number having N zeros just after the decimal point is $\{-(n+1) + a\}$ fraction.

Let us take some examples on logarithm.

Example-1:

If $log_x 625 = 4$; find the value of x.

Solution:

$log_x 625 = 4$ can be expressed in exponential form as

$$x^4 = 625$$
$$or,\ x^4 = 5^4$$
$$or,\ x = 5^{\frac{4}{4}} = 5$$

Example-2:

If $\log_{\sqrt{27}} x = -\frac{4}{3}$, find the value of x.

Solution:

Expressing $\log_{\sqrt{27}} x = -\frac{4}{3}$ in the exponential form, we get $\left(\sqrt{27}\right)^{-\frac{4}{3}} = x$

$$\text{or,}\quad x = \left(\sqrt{3^3}\right)^{-\frac{4}{3}}$$

$$\text{or,}\quad x = \left(3^{\frac{3}{2}}\right)^{-\frac{4}{3}}$$

$$\text{or,}\quad x = 3^{-2} = \frac{1}{3^2}$$

$$\text{or,}\quad x = \frac{1}{9}$$

Example-3:

If $10^x = 8$, find the value of x.

Solution:

Here $10^x = 8$ can be expressed in logarithmic form as, $log_{10} 8 = x$

Therefore, $x = log_{10} 8 = 0.9030$ (by using scientific calculator).

Example-4:

The logarithm of a number is –3.153. Find the characteristics and mantissa.

Solution:

Let $log\ N = -3.153$

$= (-3 - 0.153) = (-3 -1 +1 - 0.153) = -4 + 0.847$
$\therefore$ The characteristics is – 4 and mantissa is 0.847.

Example-5:

Find the logarithm whose logarithm is 2.4678.

Solution:

From the Anti-log Table,

For mantissa 0.467, the number = 2931

For mean difference 8, the number = 5

∴ For mantissa 0.4678, the number = (2931 + 5) = 2936.

The characteristics is 2, therefore the number must have 3 digits in the integral part.

Hence, antilog 2.4678 = 293.6

Example-6:

Find the number whose logarithm is – 2.4678.

Solution:

Let $log\ N = -2.4678$ = –2 –1 + 1 – 0.4678 = – 3 + .5322 = 3.5322

From Antilog Table,

For mantissa 0 .532, the number = 3404.

For mean difference 2, the number = 2

∴For mantissa 0.5322, the number = (3404 + 2) = 3406

The characteristic is –3, therefore the number is less than one and there must be two zeros just after the decimal point.

Hence, antilog –2.4678 = 0.003406.

Example-7:

Find the value of (i) $log_2 64$; (ii) $log_3 \frac{1}{9}$; (iii) $log_9 3$ (iv) $log_8 0.25$

Solution:

(i) Let $log_2 64 = x$

$or,\ 64 = 2^x$

$or,\ 2^6 = 2^x$

$\therefore x = 6$

(ii) Let $log_3 \frac{1}{9} = x$

$or,\ \frac{1}{9} = 3^x$

$or,\ 9^{-1} = 3^x$

$or,\ 3^{-2} = 3^x$

$\therefore x = -2$

(iii) Let $log_9 3 = x$

$or\ 3 = 9^x$

$or\ 3^1 = 3^{2x}$

$or\ 2x = 1$

(iv) Let $log_8 0.25 = x$

$or,\ 0.25 = 8^x$

$or,\ \frac{1}{4} = 2^{3x}$

$or,\ 4^{-1} = 2^{3x}$

$\therefore\ x = \frac{1}{2}$

$or,\ 2^{-2} = 2^{3x}$

$or,\ 3x = -2$

$\therefore\ x = -\,2/3$

Example-8:

Find the logarithm of the following to the base indicated in brackets.
(i) 27, (3); (ii) 64, (8); (iii) 1000, (10); (iv) 0.25, (2).

Solution:

(i) $27 = 3^3$
$\therefore log_3 27 = 3.$

(ii) $64 = 8^2$
$\therefore log_8 64 = 2.$

(iii) $1000 = 10^3$
$\therefore\ log_{10} 1000 = 3.$

(iv) $0.25 = 2^{-2}$
$\therefore\ log_2 0.25 = -2.$

Example-9:

Without using tables, evaluate

$$log_{10}\frac{41}{35} + log_{10} 70 - log_{10}\frac{41}{2} + 2\ log_{10}5$$

Solution:

$$log_{10}\left(\frac{41}{35} \times 70 \times \frac{2}{41} \times 5^2\right)$$

$= log_{10}\ 100$

$= log_{10}\ 10^2$

$= 2\ log_{10}\ 10 = 2$

Example-10:

Simplify $7\ log\frac{10}{9} - 2\ log\frac{25}{24} + 3\ log\frac{81}{80}$

Solution:

$$7\ log\frac{10}{9} - 2\ log\frac{25}{24} + 3\ log\frac{81}{80}$$

$= 7\ [log\ 10 - log\ 9] - 2[log\ 25 - log\ 24] + 3\ [log\ 81 - log\ 80]$

$= 7[(log\ 5 + log\ 2) - log\ 3^2] - 2[log\ 5^2 - (log\ 3 + log\ 2^3)] + 3[log\ 3^4 - (log5 + log2^4)]$

$= 7\ log\ 5 + 7\ log\ 2 - 14\ log\ 3 - 4\ log\ 5 + 2\ log\ 3 + 6\ log\ 2 + 12\ log\ 3 - 3\ log\ 5 - 12\ log\ 2$

$= (7 - 4 - 3)\ log\ 5 + (2 - 14 + 12)\ log\ 3 + (7 + 6 - 12)\ log\ 2$

$= log2.$

Example-11:

Find the value of $log\,\frac{75}{16} - 2\ log\,\frac{5}{9} + log\,\frac{32}{243}$, when 10 is the base of each logarithm.

Solution:

$log\,\frac{75}{16} - 2\ log\,\frac{5}{9} + log\,\frac{32}{243}$

$= [log_{10} 75 - log_{10} 16] - 2[log_{10} 5 - log_{10} 9] + [log_{10} 32 - log_{10} 243]$

$= [(log_{10} 5^2 + log_{10} 3) - log_{10} 4^2] - 2[log_{10} 5 - log_{10} 3^2] + [(log_{10} 4^2 + log_{10} 2) - log_{10} 3^5]$

$= 2\ log_{10} 5 + log_{10} 3 - 2\ log_{10} 4 - 2\ log_{10} 5 + 4\ log_{10} 3 + 2\ log_{10} 4 + log_{10} 2 - 5\ log_{10} 3$

$= log_{10} 2.$

Example-12:

Prove that,

$$(log\,\frac{3}{2}).(log\,\frac{4}{3}).(log\,\frac{5}{4}).(log\,\frac{6}{5}).(log\,\frac{7}{6}).(log\,\frac{8}{7}) = 3$$

Solution:

$$L.H.S. = \frac{\log 3 \times \log 4 \times \log 5 \times \log 6 \times \log 7 \times \log 8}{\log 2 \times \log 3 \times \log 4 \times \log 5 \times \log 6 \times \log 7}$$

$$= \frac{\log 8}{\log 2} = \frac{\log 2^3}{\log 2} = \frac{3 \log 2}{\log 2} = 3$$

Therefore, *L.H.S = R.H.S.* (Proved).

Example-13:

Solve the equation:

$log_{10}(3x+2) - 2\ log_{10} x = 1 - log_{10}(5x - 3)$

Solution:

$log_{10}(3x+2) - 2\ log_{10}\ x = 1 - log_{10}(5x - 3)$

or, $log_{10}(3x+2) + log_{10}(5x - 3) - log_{10}\ x^2 = 1$

or, $log_{10}\left[\frac{(3x+2)(5x-3)}{x^2}\right] = 1$

$\therefore \ \dfrac{(3x+2)(5x-3)}{x^2} = 10^1$

or, $15x^2 - 9x + 10x - 6 = 10x^2$

or, $5x^2 + x - 6 = 0$

or, $5x^2 + x - 6 = 0$

or, $(5x+6)(x-1) = 0$

$\therefore x = 1$ or $-^6/_5$

since x cannot be negative, $x = 1$.

Example-14:

Show that $x^{\log y - \log z} . y^{\log z - \log x} . z^{\log x - \log y} = 1$

Solution:

Let the left side = N, then multiply both sides by *log*; we have

$\log N = \log (x^{\log y - \log z}) + \log (y^{\log z - \log x}) + \log (z^{\log x - \log y})$

$= (\log y - \log z) \log x + (\log z - \log x) \log y + (\log x - \log y) \log z$

$= \log y. \log x - \log z. \log x + \log z. \log y - \log x. \log y + \log x. \log z$

$- \log y. \log z$

$= 0$

Therefore□ $N = 10^0 = 1$

Hence *L. H. S = R. H. S.* (Proved).

Example-15:

Find the value of

$\log_{27} 49$; if $\log_{10} 3 = 0.4771$ and $\log_{10} 7 = 0.8451$

Solution:

Here $\log_{27} 49$ can be written (by changing base) as

$$\frac{\log_{10} 49}{\log_{10} 27} = \frac{\log_{10} 7^2}{\log_{10} 3^3} = \frac{2\log_{10} 7}{3\log_{10} 3} = \frac{2(0.8451)}{3(0.4771)} = \frac{1.6902}{1.4313} = 1.18$$

Example-16:

Prove that $\dfrac{\log_7 243}{\log_8 3. \log_{49} 32} = 6$

Solution:

By changing all logarithms on LHS to the base 10 by using the formula, we get

$$log_7 243 = \frac{\log 243}{\log 7} = \frac{\log 3^5}{\log 7} = \frac{5 \log 3}{\log 7}$$

$$log_8 3 = \frac{\log 3}{\log 8} = \frac{\log 3}{3 \log 2}$$

$$log_{49} 32 = \frac{\log 32}{\log 49} = \frac{\log 2^5}{\log 7^2} = \frac{5 \log 2}{2 \log 7}$$

Here $\frac{\log_7 243}{\log_8 3.\log_{49} 32} = log_7 243 \div (log_8 3 \times log_{49} 32)$

$$= \frac{5 \log 3}{\log 7} \times \frac{3 \log 2}{\log 3} \times \frac{2 \log 7}{5 \log 2} = 6 \text{ (Proved)}$$

Questions for Review

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. Define logarithm. Is there any distinction between natural and common logarithm?
2. What are the fundamental rules of logarithmic operations?
3. Find the value of $log_{10} 20 + log_{10} 30 - \frac{1}{2} log_{10} 36$
4. If $log_{10}2 = 0.3010$ *and* $log_{10}3 = 0.4717$;

   find (i) $log_{10}25$; and (ii) $log_{10}4.5$
5. If $\log_{\sqrt{8}} x = 3\frac{1}{3}$; find the value of *x*.
6. Evaluate $log \frac{31}{21} + log\ 49 - log\ 62 + log\ 27 - log_{30} 87$
7. If log a = 0.589; log b = 2.856 and log c = 1.963; find the value of

   $log\left(\frac{a^4 b^{\frac{1}{3}}}{c^2}\right)$
8. Find the value of $\frac{1}{3}\ log_{10} 125 - 2\ log_{10} 4 + log_{10} 32.$
9. If $log\ 3 = 0.4771$; $log\ 2 = 0.3010$ and $log\ 7 = 0.8451$, find the value of $log \frac{48}{91}$
10. If $log_{10} [98 + \sqrt{x^2 - 12x + 36}\ ] = 2$, find the value of *x*.
11. Show that $log\ 2 + 16\ log \frac{16}{15} + 12\ log \frac{25}{24} + 7\ log \frac{81}{80} = 1.$
12. Solve $log_{10} (7x - 9)^3 + log_{10} (3x - 4)^3 = 3.$
13. Prove that $11\ log \frac{10}{9} - 3\ log \frac{25}{24} + 5\ log \frac{81}{80} = log\ 3$

Multiple Choice Questions (√ the appropriate answer)

1. If $a^x = b$, then

   (a) $\log_b x = a$ (b) $\log_a x = b$ (c) $\log_a b = x$

2. If $\log_a b = c$; then

   (a) $b^c = a$ (b) $a^c = b$ (c) $a^b = c$.

3. The value of $log_5 \left(\frac{1}{625}\right)$ is

(a) 4 (b) – 4 (c) $\frac{1}{4}$

4. The value of $\log_{\sqrt{2}} 16$ is

(a) 4 (b) 8 (c) 16.

5. If $\log_8 x = \frac{2}{3}$, then the value of x is

(a) $\frac{3}{4}$ (b) $\frac{4}{3}$ (c) 4.

6. The value of $[log \frac{3}{5} + log \frac{5}{36} + log\ 12]$ is equal to:

(a) log 5 (b) log 3 (c) 0.

7. If $log\ 2 = 0.3010$ and $5^x = 400$; then x is equal to:

(a) 2.40 (b) 3.72 (c) 4.36

8. The value of $[log \left(\frac{a^2}{bc}\right) + log \left(\frac{b^2}{ac}\right) + log \left(\frac{c^2}{ab}\right)]$ is

(a) 0 (b) 1 (c) abc

9. If $log_{10} 2x = 1$, the value of x is

(a) $\frac{1}{5}$ (b) 100 (c) 5

10. The characteristic in $log\ (6.7432 \times 10^{-5})$ is

(a) –5 (b) – 4 (c) 1

11. The Mantissa of log 3274 is .5150. The value of log 0.3274 is

(a) 0.5150 (b) 1.5150 (c) 1.5150

Lesson-2: Natural Logarithm and Antilogarithm

After studying this lesson, you should be able to

- Explain the natural logarithm;
- Explain antilogarithm;.
- Apply the principles of logarithm to solve the mathematical problems.

Nature of Natural Logarithm

arithms to the e 'e' are known as ural logarithms. value of 'e' may alculated from the eries

Logarithms to the base *'e'* are known as *natural logarithms*. The value of *'e'* may be calculated from the *'e' series*, where

$$e = 1+\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} \text{ .... } \infty$$

[Here ! is factorial, where n! = n(n-1) (n-2) .... 0!]

Hence $4! = 4 \times 3 \times 2 \times 1 \times 0!$ (since $0! = 1$)

Again, $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 \times 0!$

$= 6 \times 5!$

From *'e' series*, the value of *'e'* is 2.71828.

Let, $e^x = N$

or, $log_e N = x$

When the base of logarithm is *'e'*, it may be expressed is *ln;*

i.e., $log_e N = ln\ N.$

Again, $log_{10} N = \frac{log_e N}{log_e 10}$ *(through change of base)*

or, $log\ N = \frac{InN}{In10}$

$\therefore In\ N = log\ N \times ln\ 10$

Again, $In\ 10 = \frac{\log 10}{\log e} = \frac{1}{\log e}$

$\therefore \text{In N} = \log \text{N} \times \frac{1}{\log e}$

or, $log\ N = In\ N \times log\ e$

ng scientific ulator we can ly find the value of ased number.

Using scientific calculator we can easily find the value of *'e'* based number:

For example, $log_e 5 = 1.6094$

$log_e 0.5 = -0.6931$

$log_e 10 = 2.3025$

$$log_e\, e = In\, e = 1.$$

Let us take same examples.

Example-1:

Find the value of n, if $(1.08)^n = 3$.

Solution:

Given, $(1.08)^n = 3$

or, $In(1.08)^n = In\ 3$

or, $n\ In\ (1.08) = In\ 3$

$$n = \frac{\ln 3}{\ln(1.08)} = \frac{1.0986}{0.07696} = 14.27\ (App.)$$

Example-2:

Find the value of i, if $(1+i)^{12} = 2$

Solution:

Here $(1+i)^{12} = 2$

or, $In\ (1+i)^{12} = In\ 2$

or, $12\ In\ (1+i) = In\ 2$

or, $\text{In}\ (1+i) = \frac{\ln 2}{12} = \frac{0.6931}{12} = 0.0577$

or, $(1+i) = e^{0.0577} = 1.0594$

or, $i = 1.0594 - 1 = 0.0594$

$\therefore\ i = 0.0594$

Anti-logarithm

Let $log_a\, N = x$, then N is called the anti-logarithm of x to the base a and is written in short as $antilog_a\, x$.

Let $log_a\, N = x$, then is called the anti-logarithm of x to the base a and is written in short as $antilog_a$

If $log_a\, N = x$, then $N = antilog_a\, x$

For example, if $log\ 1000 = 3$, then $antilog\ 3 = 1000$

If $log\ 708 = 2.8500$, then $antilog\ 2.8500 = 708$.

Example-3:

Find the number whose logarithm is 1.7238

Solution:

Let the number is x

Therefore, $log\ x = 1.7238$

or, x = antilog 1.7238

$\therefore$ $x = 52.9420$ (by using calculator).

Example-4:

Find the value of (539.45×49.638)

Solution:

Let $x = 539.45 \times 49.638$

$log\ x = log\ (539.45 \times 49.638)$

$= log\ 539.45 + log\ 49.638$

$= 2.3195 + 1.6981$

or, $log\ x = 4.4276$

$\therefore x = Antilog\ 4.4276 = 26{,}776.88$

Example-5:

Solve the equation $3^x \cdot 7^{2x+1} = 11^{x+5}$

Solution:

Taking logarithm of both sides, we have

$x \log 3 + (2x+1) \log 7 = (x+5) \log 11$

or, $x \log 3 + 2x \log 7 + \log 7 = x \log 11 + 5 \log 11$

or, $x \log 3 + 2x \log 7 - x \log 11 = 5 \log 11 - \log 7$

or, $x (\log 3 + 2\log 7 - \log 11) = 5 \log 11 - \log 7$

$$\therefore x = \frac{5\log 11 - \log 7}{\log 3 + 2\log 7 - \log 11} = \frac{5.2070 - 0.8451}{0.4771 + 1.6902 - 1.0414}$$

$$= \frac{4.3619}{1.1259} = 3.87\ (App.)$$

Example-6:

Evaluate by using logarithm $\frac{61.92 \times 0.07046}{401.535}$

Solution:

Let $x = \frac{61.92 \times 0.07046}{401.535}$

Taking logarithm of both sides, we have

$log\ x = log\ 61.92 + log\ 0.07046 - log\ 401.535$

$= 1.7918 + (-1.1521) - 2.6037$

$= 1.7918 - 1.1521 - 2.6037$

$= -1.964$

$\therefore\ x = \text{antilog}\ (-1.964) = 0.01086.$

Example-7:

Evaluate by using logarithm:

(i) $\dfrac{(6.284)^3.(624)^{\frac{1}{2}}}{\sqrt[4]{0.005}}$ (ii) $\sqrt[7]{\dfrac{1}{0.8176\times 36.21}}$

Solution:

(i) Let $x = \dfrac{(6.284)^3.(624)^{\frac{1}{2}}}{\sqrt[4]{0.005}}$

Taking logarithm of both sides, we have

$$\log x = 3\log(6.284) + \frac{1}{2}\log(624) - \frac{1}{4}\log(0.005)$$

$$\text{or, } \log x = 3\,[0.7982] + \frac{1}{2}\,[2.7952] - \frac{1}{4}\,(-2.3010)$$

$$\text{or, } \log x = 2.3946 + 1.3976 - 0.5753$$

$$\text{or, } \log x = 3.7922 - 0.5753 = 3.2169$$

$$\therefore\ x = \text{antilog}\ (3.2169) = 1647.78.$$

(ii) Let $x = \sqrt[7]{\dfrac{1}{0.8176\times 36.21}}$

Taking logarithm of both sides, we have

$$\log x = \log\left[\frac{1}{0.8176\times 36.21}\right]^{\frac{1}{7}}$$

$$\text{or, } \log x = \frac{1}{7}\,[\log 1 - \log 0.8176 - \log 36.21]$$

$$\text{or, } \log x = \frac{1}{7}\,[0 - 0.0875 - 1.5588]$$

$$\text{or, } \log x = \frac{1}{7}\,[0.0875 - 1.5588]$$

$$\text{or, } \log x = \frac{1}{7}\,[-1.4713] = -0.2102$$

$$\therefore\ x = \text{antilog}\ (-0.2102) = 0.6163$$

So, $x = 0.6163$.

Example-8:

Find the value of $\frac{(435)^3.(0.056)^{\frac{1}{2}}}{(380)^4}$

Solution:

Let $x = \frac{(435)^3.(0.056)^{\frac{1}{2}}}{(380)^4}$

Taking logarithm of both sides, we have

$log\ x = 3\ log\ 435 + \frac{1}{2}\ log\ 0.056 - 4\ log\ 380$

$or,\ log\ x = 3 \times 2.6385 + \frac{1}{2} \times (-1.2518) - 4 \times 2.5798$

$or,\ log\ x = 7.9155 - 0.6259 - 10.3192$

$or,\ log\ x = -3.0296$

Hence, $x = antilog\ (-3.0296) = 0.0009341.$

Example-9:

Find the 7th root of 0.00001427

Solution:

Let $x = 0.00001427$

Taking logarithm of both sides, we have

$log\ x = \frac{1}{7}\ log\ (0.00001427)$

$or,\ log\ x = \frac{1}{7}(-4.8456)$

$or,\ log\ x = -0.6922$

$or,\ x = antilog\ (-0.6922) = 0.2031$ *(App.)*

Example-10:

Find the value using logarithm, $\frac{628.24 \times 93.536}{3.786}$

Solution:

Let $x = \frac{628.24 \times 93.536}{3.786}$

Taking logarithm of both sides we get

$log\ x = log\ 628.24 + log\ 93.536 - log\ 3.786$

or, $log\ x = (2.79813 + 1.97098 - 0.57818)$

or, $log\ x = 4.19093$

$\therefore$ $x = antilog\ 4.19093 = 15521.37.$

Example-11:

Solve $2^x.3^{2x} = 100$

Solution:

$2^x.3^{2x} = 100$

or, $\log(2^x.3^{2x}) = \log 100$

or, $x \log 2 + 2x \log 3 = \log 10^2$

or, $x(0.30103) + 2x(0.47712) = 2 \log 10$

or, $0.30103x + 0.95424x = 2$

or, $1.2552x = 2$

$\therefore$ $x = \dfrac{2}{1.25527} = 1.59328.$

Hence, $x = 1.59328.$

Example-12:

Solve $3^{2x} - 3^{x+1} + 2 = 0$

Solution:

$3^{2x} - 3^{x+1} + 2 = 0$

or, $(3^x)^2 - 3^x.3 + 2 = 0$

Let $3^x = y$

$\therefore$ $y^2 - 3y + 2 = 0$

or, $y^2 - 2y - y + 2 = 0$

or, $y(y - 2) - 1(y - 2) = 0$

or, $(y - 2)(y - 1) = 0$

Now either, $y - 2 = 0$ | or, $y - 1 = 0$

either, $y - 2 = 0$	*or,* $y - 1 = 0$
or, $y = 2$	*or,* $y = 1$
or, $3^x = 2$	*or,* $3^x = 1$
or, $\log 3^x = \log 2$	*or,* $\log 3^x = \log 1$
or, $x \log 3 = \log 2$	*or,* $x \log 3 = \log 1$
or, $x = \dfrac{\log 2}{\log 3}$	*or,* $x \log 3 = 0$
or, $x = \dfrac{0.30103}{0.47712}$	*or,* $x = 0$
or, $x = 0.6309.$	

$\therefore$ *Hence x = 0 .6309 or 0.*

Example-13:

Prove that, $7 \log \frac{15}{16} + 6 \log \frac{8}{3} + 5 \log \frac{2}{5} + \log \frac{32}{25} = \log 3$

Solution:

L.H.S. $7 \log \frac{3\times5}{2^4} + 6 \log \frac{2^3}{3} + 5 \log \frac{2}{5} + \log \frac{2^5}{5^2}$

$= 7(\log 3 + \log 5 - 4 \log 2) + 6(3 \log 2 - \log 3) + 5(\log 2 - \log 5) + (5 \log 2 - 2 \log 5)$

$= 7 \log 3 + 7 \log 5 - 28 \log 2 + 18 \log 2 - 6 \log 3 + 5 \log 2 - 5 \log 5 + 5 \log 2 - 2 \log 5$

$= \log 3$ *(Proved)*

Questions for Review

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. Find the value of $(431.96)^{26}$.

2. Find the value of $\sqrt[6]{5896.31}$.

3. Find the value of $log_2\sqrt{6} + log_2\sqrt{2/3} - log\ 10.$

4. Find the value of $log_2\sqrt{3/2} + log_2\sqrt{5/3} - log_2\sqrt{5}$.

5. Find the value of x; if $log_4 x + log_2 x = 6.$

6. Evaluate $\frac{1002.76}{12\times82}$ by using logarithm.

7. Solve $10^{4x-5}.32^x = 5^{3-x}.7^x$

8. Solve for x, if $log_x (8x - 3) - log_x 4 = 2.$

9. Evaluate $\frac{61.42\times10.70}{401.53}$

Multiple Choice Questions (√ the appropriate answer)

1. $(log_{10} 40000 - log_{10} 4)$ is equal to:

   (a) 4 (b) 1000 (c) 39996

2. If $log\ (x+1) + log\ (x-1) = log\ 3$, then x is equal to

   (a) 7 (b) 2 (c) 8

3. If $log_{10} 125 + log_{10} 8 = x$; then x is equal to

   (a) –3 (b) 3 (c) 1/3

4. The value of x satisfying $log_{32} x = 0.80$ is:

   (a) 25.6 (b) 10 (c) 16.

5. If $\frac{loga}{b-c} = \frac{logb}{c-a} = \frac{logc}{a-b}$, then the value of $a^a b^b c^c$ is

   (a) abc (b) $\frac{1}{abc}$ (c) 1.

Mathematics of Finance

3

With the help of the topics covered in this unit, students will be familiar with basic knowledge and importance of the mathematics of finance in the time value of money. The unit surveys interest, depreciation, present value and future value of money and different types of annuities followed by examples.

Blank Page

Lesson-1: Interest

After studying this lesson, you should be able to

- State the nature of interest;
- Calculate the simple interest;
- Calculate the compound interest in various situations.

Nature of Interest

When x borrows money from y, then x has to pay certain amount to y for the use of the money. The amount paid by x is called interest. The amount borrowed by x from y is called principal. The sum of the interest and principal is usually called the total amount. When interest is payable on the principal only, it is termed as simple interest. On the other hand, when interest is calculated on the amount of the previous year or period, then it is called compound interest.

When interest is payable on the principal only, it is termed as simple interest.

Calculation of Simple Interest

Let P= Principal i.e., the initial sum of money invested.

I = Interest per unit money/ per unit time.

X = Period i.e., unit of time for which the interest is calculated.

A = Amount i.e., principal plus interest accrued.

The interest on 1 unit of money for 1 unit of time = i

The interest on 1 unit of money for 'n' unit of time = ni

The interest on P unit of money for 'n' unit of time = Pni

Hence A= P + Pni = P (1+ni)

The simple interest obtained on principal (P) after n years will be

= A - P

= P (1+ni) – P = (P + Pni – P) = Pni

For example, the rate of simple interest is 10% per annum means that the interest payable on Tk.100 for one year is Tk.10, i.e., at the end of one year, total amount will be Tk.110, at the end of second year, it will be Tk.120 and so on.

Example-1:

Mr. Rahim has invested Tk.30,000 for 5 years at 10% rate of interest. What will be the simple interest and amount after 5 years?

Solution:

We know that the simple interest on principal (P) for 'n' year at a rate 'i' = Pni

Here P= 30,000 , N= 5 , i= 10%=0.10

Substituting the given values we have,

Simple Interest = 30,000 × 5 × 0.10

= Tk.15,000

Hence the required simple interest of 5 years is Tk.15,000

Amount after 5 years at simple interest, A = P (1+ni)

$= 30,000\ (1 + 5 \times 0.10)$

$= 30,000\ (1.50)$

= Tk.45,000

Calculation of Compound Interest

At the end of every period, the interest earned is added to the principal to become the principal earning interest for the next period.

If i be the rate of interest per unit per period, a principal 1 accumulates at compound interest in the following manner. At the end of every period, the interest earned is added to the principal to become the principal earning interest for the next period, For example

		Amount
Principal (P)	*1*	1
Interest for the first period	i	
Principal for the 2nd period	$(i+i)$	(1+i)
Interest for the 2nd period	$i\,(1+i)$	
Principal for the 3rd period	$(1+i)\,(1+i)$	$= (1+i)^2$
Interest for the 3rd period	$i\,(1+i)^2$	
Principal for the 4th period	$(1+i)^2\,(1+i)$	$= (1+i)^3$

And so on.

Hence the amount at the end of n period = $(1+i)^n$

Thus the amount (A) of Principal (P) at the end of n periods is,

$$A = P\,(1+i)^n$$

The fundamental formula of compound interest, namely $A = P\,(1+i)^n$ is easily adopted to logarithmic calculation, where

$$\log A = \log P + n \log (1+i)$$

Now, the compound interest $= A - P$

$= P\,(1+i)^n - P$

$= P\,[(1+i)^n - 1]$

Let P = Principal, A = the total amount, t = total interest, i = annual rate of interest, n = number of period; then the compound interest can be computed by suing the following formula, which may be changed on the basis of the number of compounding time.

Compounding Time	**Total amount**	**I = Total amount (A) – Principal amount (P)**
Weekly	$A = P\,(1 + \frac{i}{52})^{52n}$	$I = P\,[(1 + \frac{i}{52})^{52n} - 1]$
Monthly	$A = P\,(1 + \frac{i}{12})^{12n}$	$I = P\,[(1 + \frac{i}{12})^{12n} - 1]$

Compounding Time	**Total amount**	**I = Total amount (A) – Principal amount (P)**
Quarterly	$A = P\,(1 + \frac{i}{4})^{4n}$	$I = P\,[(1 + \frac{i}{4})^{4n} - 1]$
Half yearly	$A = P\,(1 + \frac{i}{2})^{2n}$	$I = P\,[(1 + \frac{i}{2})^{2n} - 1]$
Annually	$A = P\,(1 + i)^{n}$	$I = P[(1 + i)^{n} - 1]$

If the interest is *i* per unit per annum, nominal convertible *'m'* times a year; i/m is converted into the principal at the end of every such compounding time; and 1 will accumulate to $(1+i/m)^{m}$ in a year. The difference $[(1+i/m)^{m} - 1]$ a year on a principal 1 is known as the effective rate of interest per annum.

The difference $[(1+i/m)^{m} - 1]$ a year on a principal 1 is known as the effective rate of interest per annum.

A Principal *m* accumulates to $A = (1+i/m)^{nm}$ in n year at the above rate.

Example-2:

Mr. Rahim has invested Tk.30,000 for 4 years at 12% rate of interest.

1. What will be the compound interest and amount after 4 years if it is compounding (a) Yearly; or (b) Monthly?
2. Find the number of years in which the sum will double itself at annual compound interest.
3. What should be the annual compound interest rate to make the amount Tk.60,000 after 4 years?

Solution:

We are given, $P = 30{,}000$, $n = 4$ and $i = 0.12$

(1)

(a) In the case of Yearly Compounding:

Compound interest after 4 years = $P\,[(1+i)^{n} - 1]$

$= 30{,}000\,[(1 + 0.12)^{4} - 1]$

$= 30{,}000\,[(1.12)^{4} - 1]$

$= 30{,}000 \times 0.5735 = \text{Tk.}17{,}205$

Amount after 4 years, $A = P\,(1+i)^{4}$

$= 30{,}000\,(1 + 0.12)^{4}$

$= 30{,}000 \times 1.5735$

$= \text{Tk.}47,\ 205$

(b) In the case of Monthly Compounding: [then m = 12]

Compound interest after 4 years = $P\,[(1+{}^{i}/_{m})^{mn} - 1]$

$$= 30{,}000\ [(1 + \left(1+\frac{0.12}{12}\right)^{12\times4} - 1]$$

$$= 30{,}000[1.6122 - 1]$$

$$= 30{,}000 \times 0.6122 = \text{Tk.}18,\ 366$$

Amount after 4 years, A = $P\,(1+{}^{i}/_{m})^{mn}$

$$= 30{,}000\ \left(1+\frac{0.12}{12}\right)^{12\times4}$$

$$= 30{,}000\ (1.01)^{48}$$

$$= 30{,}000 \times 1.6122$$

$$= \text{Tk.}48,\ 366$$

2. Let the sum will be Tk.(30,000 × 2) = Tk.60, 000 is n years.

So, P = 30,000, A = 60,000, i = .12 and n =?

Now, $A = P\,(1+i)^n$

Or, $60{,}000 = 30{,}000\ (1 + 0.12)^n$

Or, $(1.12)^n = 60{,}000/30{,}000$

Or, $(1.12)^n = 2$

Taking logarithm both sides, we have

or, n log1.12 = log2

$$n = \frac{\log 2}{\log 1.12} = \frac{0.3010}{0.0492} = 6.12 \text{ years.}$$

Hence it will take 6.12 years for Tk.30, 000 to be doubled to Tk.60,000.

3. We have, P = 30,000; A = 60,000, n = 4 and i =?

Now $A = P\,(1+i)^n$

or, $60{,}000 = 30{,}000(1+i)^4$

or, $(1+i)^4 = 60{,}000/30{,}000$

or, $(1+i)^4 = 2$

Taking logarithm both sides, we have

or, $4\log(1+i) = \log 2$

or, $\log(1+i) = \frac{\log 2}{4} = \frac{0.3010}{4}$

or, $\log(1+i) = 0.0753$

or, $(1+i) = \text{antilog } 0.0753$

or, $(1+i) = 1.1893$

or, $i = (1.1893 - 1) = 0.1893$ or 18.93%

Hence the rate of interest should be 18.93% to make the amount Tk.60,000 after 4 years.

Calculation of compound interest with growing investment (withdrawals)

Let A invested at the beginning of the first year and an additional sum B be added to the investment in each subsequent year. No withdrawals are to be made and whose sum invested is to be allowed to accumulate at a compound rate.

Hence $A = (A_0+B/i)\,(1+i)^n - B/i$

Here, A = the sum of amount

i = the rate of interest

A_0 = invested at the beginning of the year.

B = additional sum to be added / (withdrawn) in investment.

n = number of periods.

Therefore compound interest = Total Amount (A) – Principal Amount.

Compound interest = Total Amount (A) – Principal Amount.

Let us illustrate it by the following examples.

Example – 3:

Tk.10,000 is invested at the beginning of 1999. It remains invested and, on 1st January in each subsequent year, another Tk.500 is added to it. What sum will be available on 1st January 2005 if interest is compounded each year at the rate of 5% per annum?

Solution:

We know that, $A = (A_0+B/i)\,(1+i)^n - B/i$

Here, A_0 = 10,000, B = 500, i = .05, n = 6

Substituting the values we have,

$A = (10{,}000 + 500/0.05)\,(1+ .05)^6 - 500/0.05$

$= (10{,}000 + 10{,}000)\,(1.05)^6 - 10{,}000$

$= 20{,}000 \times 1.3401 - 10{,}000$

= 26,802 – 10,000 = Tk.16, 802

Therefore, the sum of amount on 1st January 2005 is Tk.16, 802.

Example–4:

A man invests Tk.10,000 once at how and withdraws Tk.1500 at the end of each year starting at the end of the first year. How much will have left after seven years if the money is invested at 4% per annum?

Solution:

We know that, $A = (A_0 + B/i)\ (1+i)^n – B/i$

Here, $A_0 = 10{,}000$, $n = 7$, $i = 0.04$, $B = -1{,}500$

Substituting the values we have

$A = \{10{,}000 + (-1500)/0.04\}\ (1+ 0.04)^7 – (-1500/0.04$

$= (10{,}000 – 37{,}500)\ (1.04)^7 + 37{,}500$

$= (-27{,}500)\ (1.3159) + 37{,}500$

$= - 36187.25 + 37{,}500$

= Tk.1, 312.75

Drawing at this rate he will only have Tk.1, 312.75 at the end of seven years.

Questions For Review:

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. Define the following: Simple interest, Compound interest, Effective rate of interest.
2. Compare between simple interest and compound interest.
3. Mr. Asif has invested Tk.1,00,000 for 5 years at 10% rate of interest.

   a. What will be the simple interest and amount after 5 years?

   b. What will be the compound interest and amount after 5 years if interest is paid

   (i) Monthly, or (ii) Quarterly?

   c. What should be annual compound interest rate to make the amount Tk.2,00,000 after 5 years?
4. At what rate of interest an amount of investment will be thrice as much as at the end of 6 years?
5. How many years will it take at 12% interest compounding annually for Tk.6000 to grow to Tk.11, 000?
6. Tk.50,000 invested at the beginning of 1997. It remains invested and, on 1st January of each subsequent year, another Tk.5000 is added to it. What sum will be available on 1st January 2005 if interest is compounded yearly @ 10% per annum?

Multiple choice questions (√ the appropriate answers)

1. A man borrows Tk.2000 and pays back after 3 years at 10% simple interest. The amount paid by the man is:

   a) 2400 b) 2600 c) 2750
2. The difference between the simple interest and compound interest for 2years at 4% p.a. is Tk.20. The principle amount will be

   a) 12500 b) 12000 c) 13000
3. A person takes a loan of Tk.200 at 5% simple interest. He returns Tk.100 at the end of 1 year. In order to clear his dues at the end of 2 years, he will pay

   a) 110 b) 115 c) 115.50
4. Two equal amount of money are deposited in two banks, each at 15% simple interest p.a. for 3.5 years and 5years. If the difference between their interest is Tk.144, each sum is

   a) 640 b) 500 c) 720

5. In what time will a sum of money double itself at $6^1/_4$ p.a. simple interest

   a) 16years b) 12 years c) 8 years

6. A sum of money will triple itself in 15 years at simple interest with yearly rate of

   a) 12(2/3)% b) 13(1/3)% c)16(2/3)%

7. A sum was put at simple interest at a certain rate for 2 years. Had it been put at 3% higher rate, it would have fetched Tk.72 more. The sum is

   a) Tk.1600 b) Tk.1800 c) Tk.1200

8. What annual payment will discharge a debt of Tk.580 due in 5 years, the rate being 8% p.a.?

   a) 100 b) 120 c) 166.40

9. A sum of Tk.400 would become Tk.441 after 2 years if the rate of compound interest were:

   a) 5% b) 7.5% c)2.5%

10. A sum of Tk.12,000 deposited at compound interest becomes double after 5 years. After 20years it will become

    a) Tk.1, 92,00 b) Tk.1, 24,00 c) Tk.1,20,00

Lesson-2: Depreciation

After studying this lesson, you should be able to:

- Explain the nature of depreciation and depreciated value;
- Calculate the amount of depreciation under the different methods of depreciation.

Nature of Depreciation

In case of depreciation, the principal value is diminished every year by some amount, and in the subsequent period the diminished value becomes the principal value. In case of uniform decrease or depreciation, '*i*' is to be substituted by '–*i*' in the formula of future value. In that case depreciated value and accumulated depreciation is calculated by using the following formula:

In case of uniform decrease or depreciation, 'i' is to be substituted by '–i' in the formula of future value.

Depreciated value = $P\ (1\text{-}i)^n$ [Under reducing balance method]

Accumulated depreciation = $P\ [1\text{-}\ (1+i)^n]$

Where, P = Cost price of the asset

i = Rate of depreciation

n = Number of periods the asset has been depreciated.

For calculation of depreciated value and accumulated depreciation the following examples are highlighted here.

Example-1:

A machine has been purchased in 1999 at a cost of Tk.3,00,000. The machine is depreciated @8% per annum on reducing balance method. Compute-

i. What would be the depreciated value of the machine at the end of 2005?
ii. What amount should be charged as depreciation of the machine for 2006?
iii. Would it be profitable to sale the machine for Tk.1,20,000 at the end of 2007?
iv. When the depreciated value of the machine will be Tk.1,02,550?

Solution:

We are given, P = 3,00,000, i = 0.08

i. The machine has been purchased in 1999. At the end of 2005, it will be 7 years' old. Hence, the depreciated value of the machine at the end of 2005 would be,

$= P\ (1\text{-}i)^n$

$= 3{,}00{,}000\ (\ 1\text{-}\ 0.08\ )^7 = (3{,}00{,}000 \times 0.5578) =$ Tk.1,67,340.

ii. The depreciated value of the machine at the end of 2006, would be,

$= P\,(1\text{-}i)^n$

$= 3{,}00{,}000\,(1 - 0.08)^8$

$= (3{,}00{,}000 \times 0.5132) = \text{Tk.}1{,}53{,}960$

Therefore depreciation for 2006 would be:

(1,67,340 – 1,53,960) = Tk.1,41,630

iii. The depreciated value of the machine at the end of 2007 would be,

$= P\,(1\text{-}i)^n$

$= 3{,}00{,}000\,(1 - 0.08)^9$

$= 3{,}00{,}000 \times 0.4721 = \text{Tk.}1{,}41{,}630$

Hence, it would not be profitable to sale the machine for Tk.1,20,000 at the end of 2007.

iv. Let after *n* years the depreciated value of the machine would be Tk.1,02,550.

Now $3{,}00{,}000\,(1 - 0.08)^n = 1{,}02{,}550$

or, $(0.92)^n = 1{,}02{,}000/3{,}00{,}000 = 0.3418$

Taking logarithm both sides we have

or, $n \log 0.92 = \log 0.3418$

So, $n = \dfrac{\log 0.3418}{\log 0.92} = \dfrac{-0.4662}{-0.0362} = 12.88$ years.

Therefore, after 12.88 years the depreciated value of the machine would be Tk.1,02,550.

Different Methods for Calculation of Depreciation

The depreciation calculations have a significant impact on cash flows after taxes (CFAT).

The depreciation calculations have a significant impact on cash flows after taxes (CFAT). It is because a firm can legitimately deduct depreciation from its gross income to arrive at it’s before tax income. Different methods of depreciation affect tax liability, and hence the cash flows differently. Generally, there are three methods of depreciation calculations which are discussed as under.

(1) **Straight Line Method:** Under this method, depreciation charges are allocated equally over the asset’s economic life. The amount of annual depreciation charge is given by the formula:

Amount of Annual Depreciation = (Original cost – Salvage Value) ÷Economic life of an asset.

(2) **Sum-of-the-year’s-Digits Method:** In this method, the depreciation base in each year is the same as in the straight-line method - *original cost less salvage value.* However, depreciation factor changes in each year. The depreciation factor for any year is the number of useful years remaining in the life of the project taken from the beginning of the year divided by the sum of a series of numbers representing the years of service life. The depreciation factor of multiplier is calculated by the following formula:

Multiplier = $\frac{n(n+1)}{2}$ where n = Economic life of asset.

(3) Double Declining Balance Method: It is more popularly known as the twice straight line depreciation method. Under this method, the amount of depreciation to be charged is twice the straight line rate. For example, a machine which has been purchased for Tk.2,20,000 has a salvage value of Tk.20,000 and economic life of 5 years. The straight line depreciation would be Tk.40,000 per year and, therefore, annual depreciation will be $\left(\frac{40{,}000}{220000-20000}\right)$ = 20% of the depreciable value. Then the rate would be 40% under the double declining balance method. This 40% depreciation rate would be applied to the book value of the asset each year until book value equals salvage value (Tk.20,000).

Here, annual depreciation = $\left[\frac{Book\ value}{Economic\ life}\times 2\right]$.

Let us take an example to illustrate the depreciation calculations under different methods of depreciation.

Example-2:

A firm purchased a machine for Tk.4,00,000. Its useful life was 5 years and salvage value of Tk.10,000. Calculate the amount of depreciation under different methods of depreciation.

(1) Straight Line Depreciation Method

Depreciation Annual = (4,00,000-10,000)/5 = 3,90,000/5 = 78,000 (From 1^{st} year to 5^{th} Year)

(2) Sum-of-the-Years Digit Depreciation Method.

Depreciation factor/Multiplier (S) = $\frac{n(n+1)}{2} = \frac{5(5+1)}{2}$ = 5 × 3 = 15

Year	Multiplier	Depreciation (in Tk.)	
1	5/15	($^5/_{15}$ × 3,90,000) =	1,30,000
2	4/15	($^4/_{15}$ × 3,90,000) =	1,04,000
3	3/15	($^3/_{15}$ × 3,90,000) =	78,000
4	2/15	($^2/_{15}$ × 3,90,000) =	52,000
5	1/15	($^1/_{15}$ × 3,90,000) =	26,000

(3) Double Declining (reducing) Balance Depreciation Method.

Year	Total Investment	Depreciation (in Tk.)	Year-end Book value
1	3,90,000	(3,90,000/5) × 2 = 1,56,000	2,34,000
2	2,34,000	(2,34,000/5) × 2 = 93,600	1,40,400
3	1,40,400	(1,40,400/5) × 2 = 56160	84240

Year	Total Investment	Depreciation (in Tk.)	Year-end Book value
4	84,240	(84,240/5) × 2 = 33,696	50,544
5	50,544	= 50544	00

Example-3:

A machine depreciates @ 10% of its value at the beginning of the year. The machine was purchased for Tk.5,810 and the scrap value realized when sold was Tk.2,250. Find out the number of years during which the machine was in use.

Solution:

We know, $A = P(1-i)^n$

Here $A = 2{,}250$, $P = 5{,}810$, $i = 0.10$, $n = ?$

Substituting the given value we have

$$2{,}250 = 5{,}810\,(1-i)^n$$

$$\text{or, } (1 - 0.10)^n = \frac{2250}{5810}$$

$$\text{or, } (0.90)^n = 0.38726$$

Taking logarithm both sides we get

$$n \log 0.90 = \log 0.38726$$

$$\text{or, } n(-0.04576) = -0.412$$

$$\text{or } n = \frac{-0.412}{-0.04576} = 9$$

Example-4:

The life of A machine is estimated to be 10 years and the machine costs Tk.10,000. Calculate the scarp value at the end of its life; depreciation on the reducing balance method being charged @ 10% per annum.

Solution:

We know that, $A = P(1-i)^n$

Here $P = 10{,}000$, $n = 10$, $i = 0.1$

Putting the values we have,

So, $A = 10{,}000\,(1 - i)^{10}$

or, $A = 10{,}000\,(0.90)^{10}$ [By using calculator, here, A = 3486.78]

or, $\log A = \log 10{,}000 + 10 \log 0.90$

or, $\log A = 4 + 10(-0.0458)$

or, $\log A = 4 - 0.458$

or, $\log A = 3.542$

or, $A = \text{antilog} 3.542 = 3483.37$

Hence the scarp value is Tk.3,483.37

Questions for review

1. A machine has been purchased in 1995 at a cost of Tk.1,00,000. The machine is depreciated @12% p.a. on reducing balance method.

   A. What would be depreciated value of the machine at the end of 2005?

   B. What amount should be charged as depreciation of the machine for 2001?

   C. Would it be profitable to sale the machine for Tk.5,00,000 at the end of 2002?

   D. When the depreciated value of the machine will be Tk.4,20,550?

2. The value of a machine depreciates @ 10% p.a. If its present value is Tk.81000, what will be its worth after 2 years? What was the value of the machine 2 years ago?

3. A machine depreciates @ 12% p.a of its value at the beginning of a year. The machine was purchased for Tk.58,100 and the scrape value released when sold was Tk.10,000 Find out the number of years during which the machine was in use?

Multiple choice questions (√ the appropriate answers)

1. The value of a machine depreciates @ 16% p.a. If the price of the new machine is Tk.62,000, its value after 5 years will be:

   a) Tk.12400 b) Tk.25929 c) Tk.30868

2. The value of a machine depreciates @ 10% p.a. It was purchased 3years ago. If the present value is Tk.26,389.80, the purchase price of the machine was:

   a) Tk.37500 b) Tk.35600 c) Tk.36200

3) A firm purchased a machine for Tk.3,00,000. Its useful life was 6 years with salvage value of Tk.30,000. The yearly amount of depreciation using straight line method was:

   a) Tk.45, 000 b) Tk.44, 000 c) Tk.47, 000

Lesson-3: Present Value and Future Value of Money

After studying this lesson, you should be able to:

- Explain the importance of present value;
- Future value as a tool of mathematics of finance;
- Apply formula for calculating the present value;
- Future value of the money.

Introduction

The present value of a taka received after sometime will be less than a taka received today.

Generally time value of money means that the value of a sum of money received today is more then its value received after some time. Conversely, the sum of money received in future is less valuable than it is today. In other words, the present value of a taka received after sometime will be less than a taka received today.

The time value of money can also be referred to as time preference for money. The main reasons for time preference for money are to be found in the re-investment opportunities for funds which are received early. The funds so invested will earn a rate of return which will not be possible in case they are received later. The time preference for money is, therefore, expressed generally in terms of a rate of return or more popularly as a discount rate.

Nature of Present Value and Future Value

The procedure finding present values is commonly called discounting and the future values is commonly called compounding.

The concept of present value is the exact opposite of that of compound or future value. While in the later approach money invested now appreciates in value because compound interest is added, in the former approach (Present Value Approach) money is received at same future date from now and will be worth less because we have lost the corresponding interest during the period. In other words, the present value of money that will be received in the future will be less than the value of money in hand today. Thus, in contrast to the corresponding approach where we convert present value into future value, in present value approach future values are converted into present value. Given a positive rate of interest the present value of future money will always be lower. It is for this reason, therefore, that the procedure finding present values is commonly called discounting and the future values is commonly called compounding.

Let a certain amount will be received after n years; now, if the discounting rate is i, then the present value of money will be calculated by using the following formula:

Present value (PV) $= A/ (1+i)^n$

$= A (1+i)^{-n}$

Again if the different amounts of a series are received (paid) at the end of each year, than present value of money will be calculated by using the following formula:

$PV = A_1/ (1+i)^1 + A_2/(1+i)^2 + A_3/(1+i)^3 +$ ----------------$+A_n/(1+i)^n$

Where, PV = the sum of the individual present values of a separate cash flows and A_1, A_2, A_3,----------A_n refer to cash flows at the end of time periods 1, 2, 3,------- *n* respectively.

But if the different amount of a series are received / (paid) at the beginning of each year, then the present value will be computed by using the following formula:

$PV = A_1 + A_2/(1+i)^1 + A_3/(1+i)^2 +$ ------------------ $+ A_n/(1+i)^{n-1}$

On the other hand, let *A* is a present amount in corresponding to compounding rate *i*, then the future value after *nth* years will be calculated by using the following formula:

Future Value (FV) = $A(1+i)^n$

If the different amount of a series are received / (paid) at the beginning of each year, the future value after *nth* year will be computed by using following formula:

$FV = A_1(1+i)^n + A_2(1+i)^{n-1} + A_3(1+i)^{n-2} +$ ------------- $+ A_n(1+i)$

Where A_1, A_2, A_3, -----------------A_n refer to cash inflows/(outflows) at the beginning of different periods 1, 2, 3, -------------- *n* respectively.

But if the different amount of a series are received/(paid) at the end of each year, then future value after nth year will be calculated by using the following formula:

$FV = A_1(1+i)^{n-1} + A_2(1+i)^{n-2} + A_3(1+i)^{n-3} +$ ------------------- $+A_n$

The following section of this lesson contains some model application of concepts relating to the present value and future value of money.

Example-1:

Mr. Khan has following three alternatives after investing Tk.2,00,000 at now:

(a) Collecting Tk.3,00,000 after 3 years;

(b) Collecting Tk.1,40,000, Tk.1,20,000, and Tk.80,000 at the end of each year next.

(c) Collecting Tk.1,20,000, Tk.1,10,000 and Tk.80,000 at the beginning of each year next.

What alternative would be profitable for Mr. Khan if his expected rate of return is 15% p.a.?

Solution:

In case of alternative (a)

A = 3,00,000; n = 3; i = 0.15

So, Present Value (PV) = $A/(1+i)^n$

= $3,00,000/(1+0.15)^3$

= $3,00,000/(1.15)^3$

= 3,00,000/1.5209

= Tk.1,97,251.63.

In case of alternative (b)

$A_1 = 1,40,000$; $A_2 = 1,20,000$; $A_3 = 80,000$; $i = 0.15$

So, $PV = A_1/(1+i)^1 + A_2/(1+i)^2 + A_3/(1+i)^3$

$= 1,40,000/(1+0.15)^1 + 1,20,000/(1+0.15)^2 + 80,000/(1+0.15)^3$

$= 1,40,000/1.15 + 1,20,000/1.3225 + 80,000/1.5209$

$= 1,21,739.13 + 90,737.24 + 52600.43$

= Tk.2,65,076.80.

In case of alternative (c)

$A_1 = 1,20,000$; $A_2 = 1,10,000$; $A_3 = 80,000$

So, $PV = A_1 + A_2/(1+i)^1 + A_3/(1+i)^2$

$= 1,20,000 + 1,10,000/(1+0.15)^1 + 80,000/(1+0.15)^2$

$= 1,20,000 + 1,10,000/1.15 + 80,000/1.3225$

$= 1,20,000 + 95,652.17 + 60,491.49$

= Tk.2,76,143.66.

Hence, Mr. Khan should select the alternative (c) due to having higher present value of expected future cash inflows.

Example-2:

Mr. Rafiq wants to invest Tk.1,00,000, Tk.1,60,000, Tk.2,00,000 and Tk.2,40,000 at the beginning of the following 4 years respectively. If the expected rate of return is 12%, would be it possible to get Tk.10,00,000 after 4 years?

Solution:

We know that,

$\text{Future Value} = A_1 (1+i)^n + A_2 (1+i)^{n-1} + A_3 (1+i)^{n-2} + A_4 (1+i)^{n-3}$

Here $A_1 = 1,00,000$; $A_2 = 1,60,000$; $A_3 = 2,00,000$; $A_4 = 2,40,000$; $n = 4$; $i = 0.12$

Substituting the given values we have

$FV = 1,00,000 (1 + 0.12)^4 + 1,60,000 (1 + 0.12)^{4-1} + 2,00,000 (1 + 0.12)^{4-2} + 2,40,000 (1+0.12)^{4-3}$

$= 1,00,000(1.12)^4 + 1,60,000(1.12)^3 + 2,00,000(1.12)^2 + 2,40,000(1.12)^1$

$= 1,00,000(1.5735) + 1,60,000(1.4049) + 2,00,000(1.25444) + 2,40,000(1.12)$

$= 1,57,350 + 2,24,784 + 2,50,880 + 2,68,800$

= Tk.9,01,814.

Hence it will not be possible for Mr. Rafiq to get Tk.10,00,000 after 4 years from his planned investment.

Questions for Review:

These questions are designed to help you asses how far you have understood and apply the learning you have accomplished by answering (in written form) the following questions:

1. What do you mean by present value and future value of the money? Determine the present value of Tk.10,000 to be received 5 years from now, assuming 10% annual interest.
2. Find the present value of Tk.2,500 payable 4 years from now at 8% discounting (i) Quarterly (ii) Monthly.
3. Calculate the present values of the following alternatives:

Alternative	Rate	Annual flow starting after one year	Number of Years
(a)	10%	Tk.10,000	8
(b)	8%	Tk.8,000	10
(c)	14%	Tk.15,000	6

4. Mr. Nuruddin wants to invest Tk.20,000, Tk.30,000, Tk.40,000, Tk.50,000 and Tk.60,000 at the end of the following 5 years respectively. If the expected rate of return is 10%, would it be possible to get Tk.4,00,000 after 5 years?
5. Mr. Jewel has following two alternatives after inviting Tk.1,50,000 at now:

   (a) Collecting Tk.3,20,000 after 5 years.

   (b) Collecting Tk.50,000, Tk.60,000, Tk.70,000, Tk.80,000 and Tk.85,000 at the end of each year next.

   What alternative would be profitable for Mr. Nuruddin if his expected rate of return is 12% p.a.?
6. Mr. Karim owes to Mr. Rahim Tk10, 000 due in 4 years and Tk.6,000 due in seven years. Mr. Rahim agrees to pay Tk.3,000 at now, how much should have to be paid 5 years from now to settle his entire debt, assuring that money is worth 13% compounded semi-annually.

Multiple choice questions (√ the appropriate answers)

1. Tk.5000 due 5 years from now. If interest is at 4% compounding semi-annually, the amount of present value is:

   a) Tk.4202 b) Tk.4102 c) Tk.4505
2. An individual expects to receive Tk.700 after 8 years. What is the present value of the expected receipt assuming an annual compound interest rate of 8%?

   a) Tk.378 b) Tk.380 c) Tk.490
3. How much amount should be deposited now at 5% annual compounding, if the account grows Tk.500 in 10 years?

   a) Tk.301 b) Tk.307 c) Tk.320

Lesson-4: Annuity

After studying this lesson, you should be able to:

- Explain the different types of annuities,
- Calculate annuity by yourself.

Nature of Annuity

A series of uniform payments is called an annuity.

A series of uniform payments is called an annuity. In other words, an annuity is a series of payments of a fixed amount at regular intervals generally. The interval is a year, but it may be six months, or a quarter or a month.

Annuities can be divided into two classes – (1) Annuity certain and (2) Annuity contingent. In annuity certain, the payments are to be made unconditionally, for a certain or fixed number of years. In annuity contingent, the payments are to be made till the happening of some contingent even such as the death of a person, the marriage of a girl, the education of a child reaching a specified age. Life annuity is an example of annuity contingent.

Annuity certain can be divided into :(i) annuity due; and (ii) immediate annuity. When the payment of an annuity is at the beginning of each period, it is said to be an annuity due. When the payment is at the end of each period, the annuity is termed as immediate annuity.

The Present Value of an Annuity

The present value of an annuity is the sum of the present values of its installments.

The present value of an annuity is the sum of the present values of its installments. In calculating the present value of an annuity it is always customary to reckon compound interest.

Let *A* be the annuity, *V* is the present value, *i* is the rate of interest per year and *n* the number of years to continue, and then the present value of an immediate annuity is calculated by the following formula:

$$V = \frac{A}{i}\left[1 - \frac{1}{(1+i)^n}\right]$$

On the other hand the present value of an annuity due is calculated by the following formula:

$$V = \frac{A}{i}(1+i)\left[1 - \frac{1}{(1+i)^n}\right]$$

or, $V = A/i\ (1+i)\ [1 - 1/(1+i)^{n-1}]$

Let us illustrate it by two examples.

Example-1:

An investment will yield Tk.10,000 per annum for 8 years. If finance can be obtained at 7% per annum and the investment costs Tk.50,000, is it worth undertaking?

Solution:

We know that the present value of the immediate annuity would be

$$V = A/i\ [1-1/(1+i)^n\ ]$$

Here A = Tk.10,000

$i = 0.07$

$n = 8$

Substituting the given values we have

$$V = 10{,}000/0.07\ [1-1/(1+0.07)^8]$$

$$= 10{,}000/0.07\ [1-1(1.07)^8]$$

$$= 1{,}42{,}857\ [1-1/1.7182]$$

$$= 1{,}42{,}857\ [1-0.5820]$$

$$= 1{,}42{,}857 \times 0.4180$$

$$= \text{Tk.}59{,}714.23\ (\text{App})$$

Since the investment's actual cost is Tk.50,000 and the present value of the annuity is Tk.59,714.23; the investment should be made.

Example-2:

Mr. Karim can purchase a machine by paying Tk.40,000 in cash at now. He can also purchase the machine by 8 equals' yearly installments to be paid at the beginning of each year. If the interest rate is 12%, what should be amount of each installment?

Solution

Let A be the annual installment. Then Tk.40,000 is the present value of this annuity due. We are given,

$V = 40{,}000$, $n = 8$, $i = 0.12$ and $A =?$

Using the formula

$$V = \frac{A(1+i)}{i}\left[1-\frac{1}{(1+i)^n}\right]$$

$$\text{Or, } 40{,}000 = \frac{A(1+0.12)}{0.12}\left[1-\frac{1}{(1+0.12)^8}\right]$$

$$\text{Or, } 40{,}000 = A\left[\frac{1.12}{0.12}\right]\left[1-\frac{1}{2.4760}\right]$$

Or, 40,000 = A (9.3333) (1-0.4039)

Or, 40,000 = A (9.3333) (0.5961)

Or, 40,000 = A (5.5636)

Or, A = ($^{40000}/_{5.5636}$) = 7189.59

Hence the amount of each investment should be Tk.7189.59

Amount of an Annuity

Let *A* be the annuity, *i* the rate of interest per year, *n* the total time period of an annuity and *M* the future amount of annuity after *n* years, then the total amount of an immediate annuity is calculated by the following formula:

$$M = \frac{A}{i}\left[(1+i)^n - 1\right]$$

On the other hand, the total amount of an annuity due is calculated by the following formula:

$$M = \frac{A(1+i)}{i}\left[(1+i)^n - 1\right]$$

The difference between amount of installment and interest is termed as amortization.

In the repayment of a loan, it is sometimes arranged that the repayment is to be made in equal periodical installments, including repayment of principal and interest. The difference between amount of installment and interest is termed as amortization.

The following section of this lesson contains some model applications of amount of an annuity.

Example-3:

A machine costs the company Tk.98,000 and its effective life is estimated to be 12 years. If the scrap realizes Tk.3,000 only, what amount should be retained out of profits at the end of each year to accumulate at compound interest at 5% per annum?

Solution:

Let *A* be the annual installment. Evidently the amount of the annuity *A* to continue for 12 years, i.e. the balance amount to be retained = (98,000 - 3,000) = 95,000

We know that M = A/i [(1+i)^n -1]

Here, *M* = 95,000; *i* = 0.05; *n* = 12 and *A* = ?

Now putting the values we get,

95,000 = A/0.05 [(1+0.05)^{12} -1]

Or, 95,000 = A/0.05 [(1.05)^{12} -1]

Or, 95,000 = A/0.05 [1.7959 - 1]

Or, 95000 0.05 = A (0.7959)

Or, $A = \frac{4750}{0.7959} = 5968.09$

So, Tk.5968.83 should be retained out of profits at the end of each year.

Example-4:

Mr. Zahad wants to purchase a machine after 10 years when it will cost Tk.6,00,000. From now, he wants to save money for the machine and plans to deposit money into bank in 10 equal installments, the first deposit is to be made immediately. Calculate the amount of each installment reckoning compound interest at 10% p.a.

Solution:

Here the deposit pattern is an annuity due. We are given,

M = 6,00,000; i = 0.10; n = 10 and A = ?

Now using the formula

M = A/i (1+i) [$(1+i)^n$ -1]

or, 6,00,000 = A/0.10(1+0.10) [$(1+0.10)^{10}$ -1]

or, 6,00,000 = A/0.10(1.10) [$(1.10)^{10}$-1]

or, 6,00,000 = A(11) (2.5937-1)

or, 6,00,000 = A(11) (1.5937)

or, 6,00,000 = 17.5307

or, A = $\frac{600000}{17.5307}$ = 34225.67

Hence Mr. Zahad has to deposit Tk.34,225.67 in each installment at the beginning of each year.

Questions for Review

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written from) the following questions:

1. Define the following with example:

   Annuity, Annuity certain, Annuity due, Contingent annuity, Immediate annuity and Amortization.

2. Mr. Karim buys a house worth Tk.3,50,000. The contract is that Karim will pay Tk.1,00,000 immediately and the balance in 15 annual equal installments with 10% per annum compound interest. How much he has to pay annually?

3. A man wishes to have Tk.1,50,000 available in a bank account when his daughter's first year college expenses begin. How much must he deposit now at 12% compounded annually if the girl is to start in college five years from now?

4. A man retires at the age of 55 years from active service and his employer gives him pension of Tk.1,500 a year paid in half yearly installments for the rest of his life. Assuming his expectation of remaining life to be 15 years and that interest is 12% p.a. payable half yearly? What single sum is equivalent to his pension?

5. Hena borrowed Tk.10,000 to buy a refrigerator. She will amortize the loan by monthly payment of Tk.R each over a period of 3 years. Find the monthly payment if interest is 12% compounding monthly. Also find the total amount Hena will pay.

6. A firm intends to invest Tk.2,50,000 at the end of each year and to receive interest on the amounts at 10% per annum. What sum of money will be available at the end of fifth year?

7. You borrowed Tk.22,000 at 12% to be repaid over the next 6 years. Equal installment payments are required at the end of each year and these payments must be significant in amount to repay the Tk.22,000 together with providing the lender at 12% return.

   Based on these data prepare an amortization schedule.

Multiple choice questions (√ the appropriate answers)

1. A house is offered for sale for Tk.25,000. The seller agrees to accept Tk.200 at the end of each month for 8 years provided the proper down payment is made. Assuming annual interest is @ 6% compounding monthly, what should be the down payment?

   a) Tk.9781 b) Tk.9652 c) Tk.10,000

2. How much should be the deposit each year @ 5% compounding annually to accumulate Tk.1000 in 10 years from now?

   a) Tk.79.50 b) Tk.75 c) Tk.85

3. A of Tk.1000 is to be paid in 5 equal annual installment loan interest at 6% p. a. compounding annually and the 1st payment is to be made after one years. What amount is to be paid in each installment?

   a) Tk.237 b) Tk.245 c) Tk.250

4. Monir buys a house worth Tk.2,50,000. The contract is that Monir will pay Tk.1,00,000 immediately and the balance in15 annual equal installments with 10% p.a. compound interest. How much he has to pay annually?

   a) Tk.19710 b) Tk.18000 c) Tk.19550

Permutation and Combination

4

The aim of this unit is to help the learners to learn the concepts of permutation and combination. It deals with nature of permutation and combinations, basic rules of permutations and combinations, some important deduction of permutations and combinations and its application followed by examples.

Blank Page

Lesson-1: Permutation

After studying this lesson, you should be able to:

- Discuss the nature of permutations;
- Identify some important deduction of permutations;
- Explain the fundamental principles and rules of permutations;
- Highlight on some model application of permutations;

Definition of Permutation

Permutations refer to different arrangements of things from a given lot taken one or more at a time. The number of different arrangements of r things taken out of n dissimilar things is denoted by ^nP_r.

Permutations refer to different arrange-ments of things from a given lot taken one or more at a time.

For example, suppose there are three items x, y and z.

The different arrangements of these three items taking 2 items at a time are: *xy, yx, yz, zy, zx* and *xz*. Thus $^nP_r = {}^3P_2 = 6$.

Again all the arrangements of these three items taking 3 items at a time are: *xyz, xzy, yzx, yxz, zxy* and *zyx*. Thus $^nP_r = {}^3P_3 = 6$.

Hence it is clear that the number of permutations of 3 things by taking 2 or 3 items at a time is 6.

Fundamental Principles of Permutation

If one operation can be done in m different ways where it has been done in any one of these ways, and if a second operation can be done in n different ways, then the two operations together can be done in $(m\times n)$ ways.

Permutations of Things All Different

Permutations of *'n'* different things taken *'r'* at a time is denoted by ^nP_r where $r\leq n$. Here, $^nP_r = n.(n-1).(n-2)........ (n-r+1)$.

Permutations of 'n' different things taken 'r' at a time is denoted by ^nP_r.

Therefore, the first place can be filled up in n ways.

The first two places can be filled up in $n.(n-1)$ ways.

The first three places can be filled up in $n.(n-1).(n-2)$ ways.

Permutation of Things Not All Different

The number of permutation of *'n'* things taken *'r'* at a time in which k_1 elements are of one kind, k_2 elements are of a second kind, k_3 elements are of a third kind and all the rest are different is given by:

$$^nP_r = \frac{r!}{K_1!.K_2!.K_3!......K_n!}$$

Circular Permutations

The number of distinct permutations of n objects taken n at a time on a circle is (n –1)!. In considering the arrangement of keys on a chain or

The number of distinct permutations of n objects taken n at a time on a circle is (n – 1)!

beads on a necklace, two permutations are considered the same if one is obtained from the other by turning the chain or necklace over. In that case there will be $\frac{1}{2}(n-1)!$ ways of arranging the objects.

Some Important Deduction of Permutations

(i) $^nP_n = n.(n-1).(n-2)$............... to n factors

$= n.(n-1).(n-2)\ldots\ldots\ldots\ldots \{n-(n-1)\}$

$= n.(n-1).(n-2)\ldots\ldots\ldots\ldots 1$

$= n.(n-1).(n-2)\ldots\ldots\ldots\ldots 3.2.1.$

$= n!$

(ii) $^nP_{n-1} = \dfrac{n!}{\{n-(n-1)\}!}$ [since, $^nP_r = \dfrac{n!}{(n-r)!}$]

$$= \frac{n!}{\{n-n+1\}!} = \frac{n!}{1!} = n!$$

(iii) $^nP_r = n.\ ^{n-1}P_{r-1}$

or, $\dfrac{n!}{(n-r)!} = n.\ \dfrac{(n-1)!}{\{(n-1)-(r-1)\}!}$

or, $\dfrac{n!}{(n-r)!} = n.\ \dfrac{(n-1)!}{(n-r)!}$

or, $\dfrac{n!}{(n-r)!} = \dfrac{n!}{(n-r)!}$ [since, $n\,(n-1)! = n!$]

$\therefore\ ^nP_r = n.\ ^{n-1}P_{r-1}$

(iv) $^nP_r = n.(n-1).(n-2)\ldots\ldots\ldots (n-r+1)$

$$= \frac{n(n-1)(n-2)\ldots\ldots\ldots(n-r+1)(n-r)!}{(n-r)!}$$

$$= \frac{n!}{(n-r)!}$$

(v) $^nP_r = {}^{n-1}P_r + r.\ ^{n-1}P_{r-1}$

$$= \frac{(n-1)!}{(n-1-r)!} + r.\ \frac{(n-1)!}{\{(n-1)-(r-1)\}!}$$

$$= \frac{(n-1)!}{(n-1-r)!} + \frac{r.(n-1)!}{(n-r)!}$$

$$= \frac{(n-1)!}{(n-r-1)!} + \frac{r.(n-1)!}{(n-r)(n-r-1)!}$$

$$= \frac{(n-1)!}{(n-r-1)!}\left[1+\frac{r}{(n-r)}\right]$$

$$= \frac{(n-1)!}{(n-r-1)!} \times \left[\frac{n-r+r}{(n-r)}\right]$$

$$= \frac{n(n-1)!}{(n-r)(n-r-1)!}$$

$$= \frac{n!}{(n-r)!} = {}^{n}P_{r}$$

$$\therefore\ {}^{n-1}P_r + r.\,{}^{n-1}P_{r-1} = {}^{n}P_r$$

The following examples contain some model application of permutations.

Example-1:

A store has 8 regular door ways and 5 emergency doors which can be opened only from the inside. In how many ways can a person enter and leave the store?

Solution:

To enter the store, a person may choose any one of 8 different doors. Once inside he may leave by any one of (8+5) =13 doors.

$\therefore$ The total number of different ways is $(8 \times 13) = 104$.

Example-2:

There are 10 routes for going from a place Chittagong to another place Dhaka and 12 routes for going from Dhaka to a place Khulna. In how many ways can a person go from Chittagong to Khulna Via Dhaka?

Solution:

There are 10 different routes from Chittagong to another place Dhaka, the person can finish the first part of the journey in 10 different ways. And when he has done so in any one way, he will get 12 different ways to finish the second part. Thus one way of going from Chittagong to Dhaka gives rise to 12 different ways of completing the journey from Chittagong to Khulna via Dhaka.

Hence the total number of different ways of finishing both the parts of the journey as desired = (No. of ways for the 1st part $\times$ No. of ways for 2nd part) = $(10 \times 12) = 120$.

Example-3:

There are 8 men who are to be appointed as General Manager at 8 branches of a supermarket chain. In how many ways can the 8 men be assigned to the 8 branches?

Solution:

Since every re-arrangement of the 8 men will be considered as a different assignment, the number of ways will be

$$^{8}P_{8} = \frac{8!}{(8-8)!} = (8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1) = 40{,}320 \text{ ways}$$

Example-4:

Six officials of a company are to fly to a conference in Dhaka. Company policy states that no two can fly on the same plane. If there are 9 flights available, how many flight schedules can be established?

Solution:

The number of flight schedule can be established for the six officials in

$$^{9}P_{6} = \frac{9!}{(9-6)!} = \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3!}{3!} = 60480 \text{ ways}$$

Thus the total number of ways is 60480.

Example-5:

In how many ways can 3 boys and 5 girls be arranged in a row so that all the 3 boys are together?

Solution:

The 3 boys will always be kept together, so we count the 3 boys as one boy. As a result the number of persons involved to be arranged in a row is 6.

They can be arranged in 6! ways = $(6 \times 5 \times 4 \times 3 \times 2 \times 1)$ = 720 ways.

But these 3 boys themselves can be arranged in 3! ways, i.e. $(3 \times 2 \times 1)$ = 6 ways.

Hence the required number of arrangement in which the boys are together will be,

= (720×6) = 4320 ways.

Example-6:

Out of the letters *P, Q, R, x, y* and *z*, how many arrangements can be made (i) beginning with a capital; (ii) beginning and ending with a capital.

Solution:

(i) One capital letter out of given 3 capital letters can be chosen in $^{3}P_{1}$ = 3 ways. Remaining the other five letters can be arranged among themselves in 5! ways, i.e. in $(5 \times 4 \times 3 \times 2 \times 1) = 120$ ways.

Hence the total number of arrangements beginning with a capital = $(120 \times 3) = 360$.

(ii) Two capital letters out of given 3 capital letters can be chosen in $^{3}P_{2}$ = 6 ways. For each choice of these two letters, remaining four letters can be arranged in 4! ways, i.e. in $(4 \times 3 \times 2 \times 1) = 24$ ways.

Therefore the required number of arrangements beginning and ending with a capital

$= (6 \times 24) = 144$.

Example-7:

Six papers are set in an examination of which two are mathematical. In how many different orders can the papers be arranged so that (i) the two mathematical papers are together; (ii) the two mathematical papers are not consecutive.

Solution:

(i) We count the two mathematical papers as one, so that the total number of arrangement can be done in 5! ways, i.e., in $(5 \times 4 \times 3 \times 2 \times 1) = 120$ ways.

Two mathematical papers can be arranged within themselves in $2! = (2 \times 1) = 2$ ways.

Hence the required number of arrangement in which the mathematical papers are always together is $= (120 \times 2) = 240$.

(ii) Again the total number of possible arrangements is $6! = (6\times5\times4\times3\times2\times1) = 720$ ways.

Hence the total number of arrangements in which mathematical papers are not consecutive is $= (720 - 240) = 480$ ways.

Example-8:

How many different numbers of 3 digits can be formed from the digits 1, 2, 3, 4, 5 and 6, if digits are not repeated? What will happen if repetitions are allowed?

Solution:

If the repetition of digits is not allowed then the required number of arrangements is, $^{6}P_{3} = \frac{6!}{(6-3)!} = \frac{6\times5\times4\times3!}{3!}$ = 120 ways.

If the repetition of digits are allowed then the required number of arrangements is,

$= (n \times n \times n) = (6 \times 6 \times 6) = 216$ ways.

Therefore 120 and 216 different numbers can be formed respectively by repeating and not repeating digits 1, 2, 3, 4, 5 and 6.

Example-9:

How many words can be formed with the help of 3 consonants and 2 vowels, such that no two consonants are adjacent?

Solution:

Let B, C and D are three consonants and A, E are two vowels. According to the question it is represented in the following figure.

B, A, C, E, D,

The vowels A and E occupy the two positions between B, C and C, D. Each of such arrangements of consonants gives rise to two arrangements of vowels. But 3 consonants can be arranged in 3 places in 3! ways, i.e., $(3 \times 2 \times 1) = 6$ ways.

Hence the total number of arrangement is $= (2 \times 6) = 12$.

Thus the number of different words to be formed is 12.

Example-10:

How many different words can be made out of the letters of the word 'ALLAHABAD'? In how many of these with the vowels occupy the even places?

Solution:

The word 'ALLAHABAD' has 9 letters, of which 'A' occurs four times, L occurs twice and the rest all are different.

Hence the required number of permutations is,

$$= \frac{9!}{4!2!} = \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4!}{4! \times 2 \times 1} = 7560$$

The word ALLAHABAD consists of 9 letters. There are 4 even place which can be filled up by the 4 vowels in 1 way only, since all the vowels are similar (all are 'A's). Moreover the remaining 5 places can be filled up by the 5 consonants of which 2 are similar in $= \frac{5!}{2!} = \frac{5 \times 4 \times 3 \times 2!}{2!} = 60$ ways.

Hence the required number of arrangement is $(1 \times 60) = 60$.

Example-11:

In how many ways can 5 boys and 5 girls, sit at a round table so that no 2 boys are together.

Solution:

Suppose that the girls be seated first. They can sit in $(5 - 1)! = 4!$ ways, i.e., in $(4 \times 3 \times 2 \times 1) = 24$ ways.

Now since the places for the boys in between girls are fixed, the option is there for the boys to occupy the remaining 5 places. There are 5! ways, i.e., $(5 \times 4 \times 3 \times 2 \times 1) = 120$ ways for the boys to fill up the 5 places in between 5 girls seated around a table already.

Therefore the total numbers of arrangement in which both girls and boys can be seated are, $(24 \times 120) = 2880$ ways.

Questions for Review

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. Indicate how many 4 digit numbers smaller than 6,000 can be formed from the digits 2, 4, 5, 6, 8, 9?
2. Find the number of arrangements than can be made out of the letters of the word "ASSASSINATION".
3. Indicate how many 5 digit numbers can be formed from the digits 2, 3, 5, 6, 8, 9 where 6 and 9 must be included in all cases.
4. In how many ways can 6 persons formed a ring?
5. How many different arrangements can be made of all the letters of the word "ACCOUNTANTS"? In how many of them the vowels stand together?
6. In how many ways 3 boys and 5 girls be arranged in a row so that all the 5 girls, are together?
7. In how many ways can the letters of the word "EQUATION" be arranged so that the consonants may occupy only odd positions?
8. In how many ways can seven supervisors and six engineers sit for a round table discussion so that no two supervisors are setting together?
9. Find the number of permutations of the word ENGINEERING.

Lesson-2: Combinations

After studying this lesson, you should be able to:

- State the nature of combinations;
- Explain the important deductions of combinations;
- Highlight on some model applications of combinations.

Definition of Combination

bination refers to erent set of groups e out of a given without repeating element, taking one ore of them at a

Combination refers to different set of groups made out of a given lot, without repeating an element, taking one or more of them at a time. In other words, each of the groups which can be formed out of n things taking r at a time without regarding the order of things in each group is termed as combination. It is denoted by ${}^nC_r$.

For example, suppose there are three things *x, y* and *z*.

The combinations of 3 things taken 2 things at a time are: *xy, yz, zx*

Thus ${}^nC_r = {}^3C_2 = 3$.

Some Important Deductions of Combinations

(i) $${}^nC_r = \frac{n!}{r!(n-r)!}$$

Generally ${}^nC_r$ combinations would produce (${}^nC_r$ × r!) permutations; i.e., $({}^nC_r \times r!) = {}^nP_r$.

Hence, $({}^nC_r \times r!) = {}^nP_r$

$${}^nC_r = \frac{{}^nP_r}{r!} = \frac{n.(n-1).(n-2)..................(n-r+1)}{r!}$$

$${}^nC_r = \frac{n.(n-1).(n-2)..................(n-r+1).(n-r)!}{r.(n-r)!}$$

$${}^nC_r = \frac{n!}{r!(n-r)!}$$

(ii) $${}^nC_0 = \frac{n!}{0!(n-0)!} = \frac{n!}{n!} = 1 \qquad \text{[since } 0! = 1\text{]}$$

(iii) $${}^nC_1 = \frac{n!}{1!(n-1)!} = \frac{n.(n-1)!}{1!(n-1)!} = n$$

(iv) $${}^nC_n = \frac{n!}{n!(n-n)!} = \frac{n!}{n!} = 1$$

(v) $${}^nC_{n-1} = \frac{n!}{(n-1)!\{n-(n-1)\}!} = \frac{n.(n-1)!}{(n-1)!(n-n+1)!} = n$$

$$\therefore {}^nC_1 = {}^nC_{n-1}$$

(vi) $^{n}C_{r} = {}^{n}C_{n-r}$

$$= \frac{n!}{(n-r)!\{n-(n-r)\}!} = \frac{n!}{(n-r)!\,r!} = {}^{n}C_{r}$$

Therefore, $^{n}C_{r} = {}^{n}C_{n-r}$

(vii) Prove that $^{n+1}C_{r} = {}^{n}C_{r} + {}^{n}C_{r-1}$

We know that $^{n}C_{r} = \dfrac{n!}{r!(n-r)!}$

$$\therefore {}^{n}C_{r} + {}^{n}C_{r-1} = \frac{n!}{r!(n-r)!} + \frac{n!}{(r-1)!\{(n-(r-1)\}!}$$

$$= \frac{n!}{r.(r-1)!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)(n-r)!}$$

$$= \frac{n!}{(r-1)!(n-r)!}\left[\frac{1}{r} + \frac{1}{(n-r+1)}\right]$$

$$= \frac{n!}{(r-1)!(n-r)!}\left[\frac{n-r+1+r}{r.(n-r+1)}\right]$$

$$= \frac{(n+1).n!}{r.(r-1)!.(n-r+1).(n-r)!}$$

$$= \frac{(n+1)!}{r!.(n-r+1)!}$$

$$= \frac{(n+1)!}{r!.\{(n+1)-r\}!} = {}^{n+1}C_{r} \text{ (Proved).}$$

The following examples illustrate some model applications of combinations.

Example-1:

Find out the number of ways in which a cricket team consisting of 11 players can be selected from 14 players. Also find out how many of these ways (i) will include captain (ii) will not include captain?

Solution:

The numbers of ways in which 11 out of 14 players can be selected are

$$^{n}C_{r} = {}^{14}C_{11} = \frac{14!}{11!(14-11)!} = \frac{14\times 13\times 12\times 11!}{11!\times 3\times 2\times 1} = 364$$

(i) As captain is to be kept in every combination, we are to choose 10 out of the remaining 13 players. Therefore the required number of ways,

$$^{13}C_{10} = \frac{13!}{10!(13-10)!} = \frac{13\times 12\times 11\times 10!}{10!\times 3\times 2\times 1} = 286 \text{ ways}$$

(ii) In this case as captain is to be excluded, therefore, we are to choose 11 out of remaining 13 players which can be done in,

$$^{13}C_{11} = \frac{13!}{11!(13-11)!} = \frac{13\times 12\times 11!}{11!\times 2\times 1} = 78 \text{ ways.}$$

Example-2:

Out of 17 consonants and 5 vowels, how many different words can be formed each containing 3 consonants and 2 vowels?

Solution:

3 consonants can be selected out of 17 in $^{17}C_3$ ways and 2 vowels can be selected out of 5 in $^{5}C_2$ ways.

$\therefore$ The number of selections having 3 consonants and 2 vowels = $^{17}C_3 \times$ $^{5}C_2$ ways.

Each of these selections contains 5 letters which can be arranged among themselves in 5! ways. Therefore the total number of words = $^{17}C_3 \times {}^{5}C_2$ $\times$ 5!

$$= \frac{17!}{3!(17-3)!} \times \frac{5!}{2!(5-2)!} \times 5!$$

$$= \frac{17\times 16\times 15\times 14!}{3\times 2\times 1\times 14!} \times \frac{5\times 4\times 3!}{2\times 1\times 3!} \times 5.4.3.2.1 = 8,16,000.$$

Example-3:

From 6 boys and 4 girls, a committee of 6 is to be formed. In how many ways can this be done if the committee contains (i) exactly 2 girls, or (ii) at least 2 girls?

Solution:

(i) The committee of 6 is to contain 2 girls and 4 boys.

Therefore 2 girls can be selected out of 4 girls in $^{4}C_2 = \frac{4.3.2!}{2.1.2!} = 6$ ways.

The remaining 4 boys can be selected out of 6 in $^{6}C_4 = \frac{6.5.4!}{4!.2.1} = 15$ ways

Therefore the required number of ways = (6 × 15) = 90 ways.

(ii) In this case the committee of 6 can be formed in the following ways.

(a) 2 girls and 4 boys, (b) 3 girls and 3 boys and (c) 4 girls and 2 boys.

We now consider all these 3 cases:

In case of (a) the committee of 6 can be formed as explained above in $^{4}C_2 \times {}^{6}C_4$ ways.

Accordingly there are $^{4}C_3 \times {}^{6}C_3$ and $^{4}C_4 \times {}^{6}C_2$ ways of forming the committee in cases of (b) and (c) respectively.

Hence, the total number of different ways

$$= ({}^4C_2 \times {}^6C_4) + ({}^4C_3 \times {}^6C_3) + ({}^4C_4 \times {}^6C_2)$$

$$= (\frac{4!}{2!(4-2)!} \times \frac{6!}{4!(6-4)!}) + (\frac{4!}{3!(4-3)!} \times \frac{6!}{3!(6-3)!}) + (\frac{4!}{4!(4-4)!} \times \frac{6!}{2!(6-2)!})$$

$$= [(6 \times 15) + (4 \times 20) + (1 \times 15)] = (90 + 80 + 15) = 185.$$

Example-4:

In an examination, a candidate is required to answer 6 out of 12 questions which are divided into two groups each containing 6 questions and he is not permitted to attempt more than 4 questions from each group. In how many ways can he make up his choice?

Solution:

The candidate can have following three choices:

(i) 2 questions from 1st group and 4 questions from 2nd group.

(ii) 3 questions from 1st group and 3 questions from 2nd group.

(iii) 4 questions from 1st group and 2 question from 2nd group.

Now first choice can be made up in ${}^6C_2 \times {}^6C_4$ ways, second choice can be made up in ${}^6C_3 \times {}^6C_3$ ways, and third choice can be made up in ${}^6C_4 \times {}^6C_2$ ways.

Hence the total number of ways that he can make up his choice

$$= ({}^6C_2 \times {}^6C_4) + ({}^6C_3 \times {}^6C_3) + ({}^6C_4 \times {}^6C_2)$$

$$= (\frac{6!}{2!(6-2)!} \times \frac{6!}{4!(6-4)!}) + (\frac{6!}{3!(6-3)!} \times \frac{6!}{3!(6-3)!}) + (\frac{6!}{4!(6-4)!} \times \frac{6!}{2!(6-2)!})$$

$$= [(15 \times 15) + (20 \times 20) + (15 \times 15)] = (225 + 400 + 225) = 850.$$

Example-5:

The question paper of admission test in 1st years B.B.A (Hons) course in Chittagong University contains 20 questions divided into 4 groups of five questions each. In how many ways can an examinee answer 10 questions taking at least 2 questions from each group?

Solution:

The questions may be answered in the following ways:

	1st group (5)	2nd group (5)	3rd group (5)	4th group (5)
(a)	2	2	2	4
(b)	2	2	4	2
(c)	2	4	2	2
(d)	4	2	2	2
(e)	3	3	2	2

	1st group (5)	2nd group (5)	3rd group (5)	4th group (5)
(f)	3	2	3	2
(g)	2	2	3	3
(h)	2	3	2	3
(i)	2	3	3	2
(j)	3	2	2	3

For (a), the total number of ways of selecting questions from the groups is

$$= {}^5C_2 \times {}^5C_2 \times {}^5C_2 \times {}^5C_4 = 5000$$

Similarly,

For (b) $= {}^5C_2 \times {}^5C_2 \times {}^5C_4 \times {}^5C_2 = 5000$

For (c) $= {}^5C_2 \times {}^5C_4 \times {}^5C_2 \times {}^5C_2 = 5000$

For (d) $= {}^5C_4 \times {}^5C_2 \times {}^5C_2 \times {}^5C_2 = 5000$

For (e) $= {}^5C_3 \times {}^5C_3 \times {}^5C_2 \times {}^5C_2 = 10000$

For (f) $= {}^5C_3 \times {}^5C_2 \times {}^5C_3 \times {}^5C_2 = 10000$

For (g) $= {}^5C_2 \times {}^5C_2 \times {}^5C_3 \times {}^5C_3 = 10000$

For (h) $= {}^5C_2 \times {}^5C_3 \times {}^5C_2 \times {}^5C_3 = 10000$

For (i) $= {}^5C_2 \times {}^5C_3 \times {}^5C_3 \times {}^5C_2 = 10000$

For (j) $= {}^5C_3 \times {}^5C_2 \times {}^5C_2 \times {}^5C_3 = 10000$

Hence the total number of ways

= (5000 + 5000 + 5000 + 5000 + 10000 + 10000 + 10000 + 10000 + 10000 + 10000)

= 80,000.

Example-6:

A cricket team consisting of 11 players is to be formed from 16 players of whom 4 can be bowlers and 2 can keep wicket and the rest can neither be bowler nor keep wicket. In how many different ways can a team be formed so that the teams contain (i) exactly 3 bowlers and 1 wicket keeper, (ii) at least 3 bowlers and at least 1 wicket keeper?

Solution:

(i) A cricket team of 11 is to be formed with exactly 3 bowlers and 1 wicket keeper.

3 bowlers can be selected out of 4 in ${}^{4}C_{3}$ ways, 1 wicket keeper can be selected out of 2 in ${}^{2}C_{1}$ ways and the other 7 players can be selected from the remaining 10 players in ${}^{10}C_{7}$ ways.

Hence the total number of ways in which the cricket team can be formed

$= {}^{4}C_{3} \times {}^{2}C_{1} \times {}^{10}C_{7}$

$= \dfrac{4!}{3!(4-3)!} \times \dfrac{2!}{1!(2-1)!} \times \dfrac{10!}{7!(10-7)!}$

$= (4 \times 2 \times 120) = 960.$

(ii) Since at least 3 bowlers and at least one wicket keeper is to be included in the cricket team of 11 players, the team can be formed by choosing.

(a) 3 bowlers, 1 wicket keeper and 7 other players.

(b) 3 bowlers, 2 wicket keeper and 6 other players.

(c) 4 bowlers, 1 wicket keeper and 6 other players.

(d) 4 bowlers, 2 wicket keeper and 5 other players.

We now consider all these 4 cases.

(a) 3 bowlers, 1 wicket keeper and 7 other players can be selected in

$= {}^{4}C_{3} \times {}^{2}C_{1} \times {}^{10}C_{7} = \dfrac{4!}{3!(4-3)!} \times \dfrac{2!}{1!(2-1)!} \times \dfrac{10!}{7!(10-7)!}$

$= (4 \times 2 \times 120) = 960$ ways.

(b) 3 bowlers, 2 wicket keeper and 6 other players can be selected in

$= {}^{4}C_{3} \times {}^{2}C_{2} \times {}^{10}C_{6} = (4 \times 1 \times 210) = 840$ ways.

(c) 4 bowlers, 1 wicket keeper and 6 other players can be selected in

$= {}^{4}C_{4} \times {}^{2}C_{1} \times {}^{10}C_{6} = (1 \times 2 \times 210) = 420$ ways

(d) 4 bowlers, 2 wicket keeper and 5 other players can be selected in

$= {}^{4}C_{4} \times {}^{2}C_{2} \times {}^{10}C_{5} = (1 \times 1 \times 252) = 252$ ways

Therefore, the total number of ways

$= (960 + 840 + 420 + 252) = 2472.$

Questions for Review

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. A committee of 5 is to be formed from 14 students. How many different ways this can be done so as always to (i) include 2 particular students; and (ii) exclude 3 particular students?
2. A question paper contains six questions, each having an alternative. In how many ways can an examinee answer one or more questions?
3. A committee consists of 5 members is to be formed out of 6 men and 4 women. How many types of committees can be formed so that at least 2 women are always there?
4. Out of 10 consonants and 4 vowels, how many words can be formed each containing 3 consonants and 2 vowels?
5. From 6 boys and 4 girls, 5 are to be selected for admission into a particular course. In how many ways can this be done if there must be exactly 2 girls?
6. In how many ways a committee of 5 members can be formed out of 8 professors? How often will each professor be selected? If one particular professor is always included, what will be the number of ways? In how many ways the committee can be formed if one particular professor is always excluded?

Equations 5

The aim of this unit is to equip the learners with the concept of equations. The principal foci of this unit are degree of an equation, inequalities, quadratic equations, simultaneous linear equations, graphical equation and their applications in solving business problems followed by examples.

Blank Page

Lesson 1: Equation and Identity

After studying this lesson, you should be able to:

- Explain the nature and characteristics of equations;
- Explain the nature and characteristics of identities;
- Solve the equations;
- Solve the inequalities.

Introduction

Many applications of mathematics involve solving equation. In this lesson we will discuss the equation, identities and uses of equations.

Equation

An equation is a statement which says that two quantities are equal to each other. An equation consists of two expressions with a '=' sign between them. In other words, if two sides of an equality are equal only for particular value of the unknown quantity or quantities involved, then the equality is called an equation.

An equation is a statement which says that two quantities are equal to each other.

For example, $4x = 8$ is true only for $x = 2$. Hence, it is an equation.

An equation which does not contain any variable is either a true statement, such as $2 + 3 = 5$, or a false statement, such as $3 + 5 = 12$. If an equation contains a variable, the solution set of the equation is the set of those values for the variable which gives a true statement when substituted into the equation.

For example, the solution set of $y^2 = 4$ is $(-2, 2)$, because $(-2)^2 = 4$, and $2^2 = 4$, but $y^2 \neq 4$ if y is any number other than –2 or 2.

Identities

The equations signify relation between two algebraic expressions symbolized by the sign of equality. If two sides of an equality are equal for all values of the unknown quantity or quantities involved, then the equality is called an identity.

If two sides of an equality are equal for all values of the unknown quantity or quantities involved, then the equality is called an identity.

For example, $x^2 - y^2 = (x + y)(x - y)$ is an identity.

We can prove that identities hold true for whatever are values of the variables substituted in these. It we use x = 2 and y = 3 in the above identity, we have $(2)^2 - (3)^2\ (2 + 3)(2 - 3)$

or, $4 - 9\ (5)(-1)$

or, $-5 = -5$

Again, by substituting the values of x = – 4 and y = – 6, we have

$(-4)^2 - (-6)^2 = (-4 - 6)(-4 + 6)$

or, $16 - 36 = (-10)(2)$

or, $-20 = -20$

Hence, identities hold true for whatever value is put for variables.

Derived Identities

Derived identities are the identities derived by transposing the values in the basic identities and are very useful in tackling some problems in mathematics.

Derived identities are the identities derived by transposing the values in the basic identities and are very useful in tackling some problems in mathematics. For example,

(1) Identity → $(x + y)^2 = x^2 + 2xy + y^2$ .... (i)
Derived Identities $x^2 + y^2 = (x + y)^2 - 2xy$
and $2xy = (x + y)^2 - (x^2 + y^2)$

(2) Identity → $(x - y)^2 = x^2 - 2xy - y^2$ .... (ii)
Derived Identities $x^2 + y^2 = (x - y)^2 + 2xy$
and $2xy = x^2 + y^2 - (x - y)^2$

By adding (i) and (ii)

$$(x + y)^2 + (x - y)^2 = 2(x^2 + y^2) \quad \text{............................ (iii)}$$

By substituting (ii) from (i), we get

$$(x + y)^2 - (x - y)^2 = 4xy \quad \text{.................................. (iv)}$$

By dividing both (i) and (ii) by 4 and then subtracting (ii) from (i), we have $[(x + y)^2 / 4] - [(x - y)^2]$

The following section of this lesson contains some model applications of equations.

Example–1:

Solve, $\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}} = \dfrac{1}{3}$

Solution:

$$\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}} = \frac{1}{3}$$

By cross multiplying, we have

$$3\sqrt{1+x}-3\sqrt{1-x} = \sqrt{1+x}+\sqrt{1-x}$$

$$\text{or, } 3\sqrt{1+x}-3\sqrt{1+x} = \sqrt{1-x}+\sqrt{1-x}$$

$$\text{or, } 2\sqrt{1+x} = 4\sqrt{1-x}$$

Squaring both sides, we have,

$4(1 + x) = 16 (1 - x)$

or, $4 + 4x = 16 - 16x$

Transposing the term 16x and 4

$4x + 16x = 16 - 4$

or, $20x = 12$

or, $x = \frac{3}{5}$

Therefore, $x = \frac{3}{5}$ is the solution of the given equation.

Example – 2:

The sum of two numbers is 45 and their ratio is 7:8. Find the numbers.

Solution:

Let, one of the numbers be x

The other number is (45 – x)

Using the given information, we get, $\frac{x}{45 - x} = \frac{7}{8}$

By cross multiplication, we get

$8x = 7(45 - x)$

or, $8x = 315 - 7x$

Transposing the term –7x, we have

$8x + 7x = 315$

or, $15x = 315$

or, $x = \frac{315}{15} = 21$

Hence, the one number is 21 and the other number is (45 – 21) = 24.

Example – 3:

The ages of a mother and a daughter are 31 and 7 years respectively. In how many years will the mother's age be $\frac{3}{2}$ times that of the daughter?

Solution:

Let the required number of years be *x*.

Mother's age after *x* years = 31 + x

Daughter's age after *x* years = 7 + x

Using the given information, we get

$31 + x = \frac{3}{2}(7 + x)$

or, $31 + x = \frac{21 + 3x}{2}$

By cross multiplication, we have

$2(31 + x) = 21 + 3x$

or, $62 + 2x = 21 + 3x$

Transposing the terms 3x and 62

$2x - 3x = 21 - 62$

or, $-x = -41$

or, $x = 41$

Hence, in 41 years, the mother's age will be $\frac{3}{2}$ times age of the daughter's.

Example–4:

The distance between two stations is 340 km. Two trains start at the same time from these two stations on parallel tracks to cross each other. The speed of one train is greater than that of other by 5 km / hr. If the distance between the two trains after 2 hours of their start is 30 km, find the speed of each trains?

Solution:

Let the speed of the first train be x km / hr.

Then the speed of the second train be $(x + 5)$ km / hr.

Distance covered by the first train in 2 hrs = $2x$ km.

Distance covered by the second train in 2 hrs = $2(x + 5)$ km = $(2x + 10)$ km.

Since, both the trains are in opposite directions.

Total distance between the two stations = 340 km.

Distance between the two trains after 2 hours = 30 km.

Therefore, distance covered by two trains in two hours from opposite directions

$= (340 - 30) = 310$ km

$\therefore 2x + (2x + 10) = 310$

or, $4x = 300$

$x = 75$

Hence, speed of the first train = 75 km / hr and the speed of the second train = 80 km / hr.

Questions For Review:

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. What is an equation and inequalities? Mention the characteristics of equation.
2. What is the difference between identity and equation? Give examples.
3. Solve the following equations:

   (i) x (x + 1) + 72 / x (x+1) = 18

   (ii) $x^2 - 6x + 9 = 4\ \sqrt{x^2 - 6x + 6}$

   (iii) $\sqrt{\frac{x}{x+6}} + \sqrt{\frac{x+6}{x}} = \frac{25}{12}$
4. Wasifa's mother is four times as old as Wasifa. After five years, her mother will be three times as old as she will be then, what are their present ages?
5. A steamer goes downstream and covers the distance between two parties in 4 hours while it covers the same distance upstream in 5 hours. If the speed of the stream is 21 cm/per hour, find the speed of the steamer in still water.
6. Three prizes are to be distributed in a quiz contest. The value of the 2^{nd} prize is five-sixths the value of the first prize and the value of the third prize is four fifths that of the second prize. If the total value of the three prizes is Tk.15,000, find the value of each prize.
7. The price of two cows and five horses is Tk.68,000. If the price of horse exceeds that of a cow by Tk.800, find the price of each.

Multiple choice questions (√ the appropriate answers)

1. The difference between two numbers is 9 and the difference between the squres is 207. The numbers are:

   a) 17, 8 b) 16, 7 c) 23, 14
2. The ratio of two numbers is 3 : 5. If each is increased by 10 the ratio becomes 5 : 7, the numbers are:

   a) 6, 10 b) 15, 25 c) 9, 15
3. The sum of three numbers is 102. If the ratio between the first and the second be 2 : 3 and that of between the second and the third be 5 : 3, the second number is

   a) 30 b) 45 c) 27
4. Taka 49 were divided among 150 children. Each girl got 50 paisa and each boy got 25 paisa. How much boys were there?

   a) 104 b) 105 c) 102
5. If $2x + 3x = 5$ and $x = -2$, then the value of y is:

   a) 1/3 b) 1 c) 3

Lesson-2: Inequality

After studying this lesson, you should be able to:

- Describe the nature of inequalities;
- Explain the properties of inequality;
- Solve the inequalities.

Nature of Inequality

Relationship of two expressions with an inequality sign between them is called inequality.

Relationship of two expressions with an inequality sign (≤ or ≥, < or >) between them is called inequality. For example,

$x > y \rightarrow$ "x is greater than y"

$x < y \rightarrow$ "x is smaller than y"

$x \ngtr y \rightarrow$ "x is not greater than y"

$x \nless y \rightarrow$ "x is not smaller than y"

$x \leq y \rightarrow$ "x is smaller than or equal to y"

$x \geq y \rightarrow$ "x is greater than or equal to y"

Properties of Inequalities

The fundamental properties of inequalities are as follows:

(a) Order Axioms: If x and b are only elements, then

(i) One and only one of the following is true:

$x = b, x < y$ and $x > y$

(ii) If $x < y$ and $y < z,$ then $y < c$

(iii) If $x < y$ and $x < z,$ then $xz < yz$

Since, '$x > y$' and '$y < x$' are the same statements, the above axioms can be replaced in terms of '$x > y$'.

As shown earlier sometimes equality signs are combined with inequality signs $x < y$ means $x = y$ or $x < y$.

Again $x < y$ means x is not less than y and that means either $x > y$.

So, $x < y$ means $y \leq x$.

We also say that x is positive when $x \geq 0$ and x is negative, when $x < 0$.

(b) Operation Axioms:

All equals may be add or subtracted from both sides of inequalities and the inequality is preserved.

(i) All equals may be add or subtracted from both sides of inequalities and the inequality is preserved.

For example, if $5x - 9 < 12$

We may add 5 to both sides and we get

$5x - 9 + 5 < 12 + 5$

or, $5x - 4 < 17$

Any term in an inequality can be moved from one side to the other provided that its sign is changed. For example, if $5x - 4 < 17$.

or, $5x < 17 + 4$

And, again if $x - z > y$

or, $x > y + z$

(ii) Both sides of an inequality may be multiplied or divided by a positive number and the inequality is preserved. For example, if $12x < 36$.

Both sides of an inequality may be multiplied or divided by a positive number and the inequality is preserved.

After Multiplying the both sides of inequality by 5, we get

$12x \times 5 < 36 \times 5$

or, $60x < 180$

Again, after dividing both sides of inequality by 3, we get

$12x \div 3 < 36 \div 3$

or, $4x < 12$

(iii) Both sides on an inequality are multiplied or divided by a negative number and the direction of the inequality is reversed. For example, if $7x < 40$, (where $x = 4$).

Both sides on an inequality are multiplied or divided by a negative number and the direction of the inequality is reversed.

By multiplying both sides of the inequality by –5, we have

$7x \times (-5) > 40 \times (-5)$ [Note that inequality sign has been changed from $<$ to $>$]

or, $-35x > -200$

This is because when $x = 4$, the inequality $-35x = -140$ is greater than –200.

(iv) An inequality can by converted into an equation:

If $x > y$ then $x = y + p$

Where p is the positive real number (i.e. $p > 0$)

If $z > m$, then we write

$z = m + q$, where $q > 0$

Hence, $x, z = (y + p)(m + q) = ym + yq + pm + pq$

Now p and q are positive. If in addition y and m are positive, then every term on the right-hand side is also positive so that

$x. z > y. m$

(v) If $\frac{x}{z} > \frac{y}{m}$, then $\frac{z}{x} < \frac{m}{y}$

(vi) If $x < y$, then $-x > -y$

(vii) Now if $x_1 > y_1, x_2 > y_2, x_3 > y_3$ ----------- $x_n > y_n$,

then $x_1 + x_2 + x_3 +$ ----------- $+x_n > y_1 + y_2 + y_3 +$ --------- $+y_n$

and $x_1. x_2. x_3$ --------- $x_n > y_1. y_2. y_3$ ----------- y_n

(viii) If $x > y$ and $n > 0$ then $x^n > y^n$ and $\frac{1}{x^n} < \frac{1}{y^n}$

The following section of this lesson contains some applications of inequality.

Example-1:

Solve: $3[4x - 5(2x - 3)] \leq 7 - 2\ [x + 3\ (4 - x)]$

Solution:

$3[4x - 5(2x - 3)] \leq 7 - 2\ [x + 3\ (4 - x)]$

or, $3[4x - 10x - 15)] \leq 7 - 2\ [x + 12 - 3x)]$

or, $12x - 30x - 45 \leq 7 - 2x + 24 + 6x$

Transposing both sides we have

or, $12x - 30x + 2x \leq 6x - 24 - 45$

or, $-12x \leq -62$

Multiplying both sides by –1, we get

or, $22x \geq 62$

or, $x \geq \frac{31}{11}$

Example-2:

Solve: $5x - 2(3x - 4) > 4[2x - 3\ (1 - 3x)]$

Solution:

$5x - 2(3x - 4) > 4[2x - 3\ (1 - 3x)]$

or, $5x - 6x - 8) > 8x - 12 + 36x$

or, $5x - 6x - 8x - 36x > -12 - 8$

or, $-45x > -20$,

Multiplying both side we get –1,

or, $45x < 20$

or, $x < \frac{20}{45}$

i.e. $x < \frac{4}{9}$

Questions for review

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. Solve the following inequalities:

   (i) $3x - 2 < 4 + 6x$

   (ii) $2x - 3(4 - x) < 7 - 4(1 - 2x)$

   (iii) $2x - 3 + 4(5 - 3x) \geq 4x - 19$

   (iv) $3 - [5x + 11 - 2x(3 + 2)] \leq 11 - 3x[x - 5(3 - x)]$

   (v) $[(5x - 7) / (2x - 3)] - 2(3 - 4x) < 0$

2. Solve each inequality with a sign graph

   (i) $(x + 3)(x + 4)^2 < 0$

   (ii) $(x - 1)(x - 2)(x - 3) < 0$

Multiple choice questions (√ the appropriate answers)

1. If $x + 2y \leq 3$, $x > 0$ and $y > 0$, then one of the solution is

   a) $x = -1, y = 2$ b) $x = 1, y = 1$ c) $x = 2, y = 1$

2. The system of linear in equalities $x + y \leq 0$, $x \geq 0$ and $y \geq 0$, has

   a) exactly 1 solution b) 3 solutions c) no solution

3. The value of x from linear inequality $4x - 7 > 6x + 5$

   a) $x < -6$ b) $x < -8$ c) $x < -4$

Lesson-3: Degree of an Equation

After studying this lesson, you should be able to:

- Explain the nature of degree of an equation;
- Solve the simultaneous linear equation.

Nature of Degree of an Equation

An ordinary equation involving only the first power of the unknown quantity is called 'simple' or 'linear' equation or equation of the first degree.

An equation involving only one unknown quantity is called ordinary equation. An ordinary equation involving only the first power of the unknown quantity is called 'simple' or 'linear' equation or equation of the first degree. When the highest power of the unknown quantity x is 2, it is called 'quadratic or the second degree equation; when the highest power of the unknown quantity x is 3, the equation is termed as 'cubic' or the third degree equation. When the highest power of x is 4, the equation is called 'biquadratic or the fourth degree equation.

For example,

$2x + 18 = y \rightarrow$ Linear equation
$2x^2 + 5x + 7 = 0 \rightarrow$ Quadratic equation
$x^3 + 5x^2 + 3x + 9 = 0 \rightarrow$ Cubic equation
$x^4 + 10x^3 + 5x^2 + 2x + 10 = 35 \rightarrow$ Biquadratic equation

If an equation in x is unaltered by changing x to $\frac{1}{x}$, it is known as a reciprocal equation.

An equation in which the variable occurs as indices or exponents is called an exponential equation.

For example, $3^x = 21$, $81^x = 9^{x+4}$ etc. are called exponential equations.

If more than one unknown quantity are involved, the number in independent equations required for solution is equal to the number of the unknown quantities. Such set of linear equations is called simultaneous linear equations.

The following section of this lesson contains some model applications.

Example-1:

Solve $4x^4 – 16x^3 + 23x^2 – 16x + 4 = 0$

Solution:

$4x^4 – 16x^3 + 23x^2 – 16x + 4 = 0$

Rearranging the terms, we have

$4x^4 – 4 – 16x^3 – 16x + 23x^2 = 0$

Dividing both sides by x^2 we have,

$$4x^2 + \frac{4}{x^2} - 16x - \frac{16}{x} + 23 = 0$$

or, $4\left(x^2+\frac{1}{x^2}\right)-16\left(x+\frac{1}{x}\right)+23=0$

or, $4\left[\left(x+\frac{1}{x}\right)^2-2.x.\frac{1}{x}\right]-16\left(x+\frac{1}{x}\right)+23=0$

Putting y for $x+\frac{1}{x}$, we have

$4\,(y^2-2)-16\,(y)+23=0$

or, $4y^2-8-16y+23=0$

or, $4y^2-16y+15=0$

or, $4y^2-10y-6y+15=0$

or, $2y\,(2y-5)-3\,(2y-5)=0$

or, $(2y-5)\,(2y-3)=0$

either, $2y-5=0$ or, $2y-3=0$

or, $2y=5$ or, $2y=3$

or, $y=\frac{5}{2}$ or, $y=\frac{3}{2}$

When, $y=\frac{5}{2}$, then $x+\frac{1}{x}=\frac{5}{2}$

or, $\frac{x^2+1}{x}=\frac{5}{2}$

or, $2x^2+2=5x$

or, $2x^2-4x-x+2=0$

or, $2x\,(x-2)-1\,(x-2)=0$

or, $(2x-1)\,(x-2)=0$

either, $2x-1=0$ or, $x-2=0$

or, $2x=1$ $\therefore x=2$

$\therefore x=\frac{1}{2}$

When, $y=\frac{3}{2}$, then $x+\frac{1}{x}=\frac{3}{2}$

$\frac{x^2+1}{x}=\frac{3}{2}$

or, $2x^2+2=3x$

or, $2x^2-3x+2=0$

We know that $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ (Here, a = 2, b = –3, c = 2)

$$x = \frac{3 \pm \sqrt{9-16}}{4} = \frac{3 \pm \sqrt{-7}}{4} = \frac{3 \pm \sqrt{7}i}{4} \quad \left(Here, \;\; i = \sqrt{-1}\right)$$

Hence, $x = 2, \; \frac{1}{2}$ or, $\frac{3 \pm \sqrt{7}i}{4}$

Example-2:

Solve $x + \frac{4}{y} = 1$

$$y + \frac{4}{x} = 25$$

Solution:

$x + \frac{4}{y} = 1$ .... (i)

$y + \frac{4}{x} = 25$ .... (ii)

From equation (i) $x + \frac{4}{y} = 1$

or, $\frac{xy+4}{y} = 1$

or, $xy + 4 = y$ .... (iii)

From equation (ii) $y + \frac{4}{x} = 25$

or, $\frac{xy+4}{x} = 25$

or, $xy + 4 = 25x$ .... (iv)

Subtracting the equation (iii) from equation (iv)

$0 = 25x - y$

or, $y = 25x$

Putting the value of y in equation (iii), we get

$x(25x) + 4 = 25x$

or, $25x^2 + 4 = 25x$

or, $25x^2 - 25x + 4 = 0$

or, $25x^2 - 20x - 5x + 4 = 0$

or, $5x(5x - 4) - 1(5x - 4) = 0$

or, $(5x - 4)(5x - 1) = 0$

either, $5x - 4 = 0$ or, $5x - 1 = 0$

or, $5x = 4$ or, $5x = 1$

so, $x = \frac{4}{5}$ so, $x = \frac{1}{5}$

Now $x = \frac{4}{5}$, then $y = 25 \times \frac{4}{5} = 20$

$x = \frac{1}{5}$, then $y = 25 \times \frac{1}{5} = 5$

Thus, $x = \frac{4}{5}$, $x = \frac{1}{5}$

$y = 20$, $y = 5$

Example-3:

Find the solution of the system of equations.

$4x - 3y + z = 1$ .... (1)

$2x - y + 2z = 6$ .... (2)

$3x + 4y - 4z = -1$ .... (3)

Solution:

Let us first eliminate z:

We rewrite: $2 \times (1)$: $8x - 6y + 2z = 2$

$\underline{(2): 2x - y + 2z = 6}$

Subtraction: $6x - 5y = -4$ .... (4)

Now we rewrite: (3) : $3x + 4y - 4z = -1$

$\underline{2 \times (2): 4x - 2y + 4z = 12}$

Addition: $7x + 2y = 11$ .... (5)

Let us now eliminate y from (4) and (5); i.e.

$5 \times (5)$: $35x - 10y = 55$

$\underline{2 \times (4) : 12x - 10y = -8}$

Addition: $47x = 47$

So, $x = 1$.

Now, substituting the value of x in (5), we get the value of y, i.e.

$7 (1) + 2y = 11$

or, $2y = 11 - 7$

or, $2y = 4$

so, $y = 2$

Now, substituting the value of x and y in (1), we have:

$4 (1) - 3 (2) + z = 1$

or, $z = 1 - 4 + 6$

or, $z = 3$

Therefore the solution is: $x = 1$, $y = 2$ and $z = 3$.

Questions for Review:

These questions are designed to help you assess how far you have understood and apply the learning you have accomplished by answering (in written form) the following questions:

1. Explain the nature of degree of an equation.
2. Solve the following equations:

    (i) $x^4 - 3x^3 - 2x^2 - 3x + 1 = 0$

    (ii) $\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} = \frac{3}{2}$

    $x - y = 3$

    (iii) $3x + 7 - 5z = 0$

    $7x - 3y - 9z = 0$

    $x^2 + 2y^2 + 3z^2 = 23$

3. Solve

    (i) $x^3 + y^3 = 4914$

    $x + y = 18$

    (ii) $27^x = a^y$

    $81^y = 243.3^x$

Multiple choice questions (√ the appropriate answers)

1. The solution of the simultaneous linear equation $\frac{2x}{3} - \frac{y}{2} = -\frac{1}{6}$ and $\frac{x}{2} - \frac{2y}{3} = 3$

    a) $x = 2,\ y = 3$ b) $x = -2,\ y = 3$ c) $x = 2,\ y = -3$

2. The solution of the system of simultaneous linear equation $4x - 3y = 7$ and $7x + 5y = 2$ is:

    a) $x = -1,\ y = 1$ b) $x = 1,\ y = -1$ c) $x = 1,\ y = 1$

3. If $7x + 9y = 85$ and $4x + 5y = 48$, then

    a) $x = 7,\ y = 4$ b) $x = \frac{40}{7},\ y = 5$ c) $x = 37,\ y = 8$

4. The solution of the pair of equations $\frac{3}{x} + \frac{6}{y} = 12$ and $\frac{4}{x} + \frac{12}{y} = 44$ is:

    a) $x = \frac{1}{-2},\ y = 3$ b) $x = -2,\ y = 3$ c) $x = -\frac{1}{2},\ y = \frac{1}{3}$

Lesson-4: Graphical Equation

After studying this lesson, you should be able to:

- State the nature of graph;
- Solve the equation with the help of graph.

Graphical Equation

In this lesson, we will discuss about graphical equations with the help of following practical examples.

Example–1:

Solve the equation graphically, $2x+5y = 12$ and $y-x = 1$

Solution:

$2x+5y = 12$ ....(i)

$y - x = 1$ ....(ii)

From equation (i), $5y = 12 - 2x$.

$$\text{So, } y = \frac{12 - 2x}{5}$$

So for x = 0, 1, 2 and 3; y = 2.4, 2, 1.6 and 1.2 respectively.

Plotting the points (0, 2, 4), (1, 2), (2, 1.6), (2, 1.6), (3, 1.2) and jointing them, we obtain graph of 2x + 5y = 12, which is AB in the following figure - 5.1.

Again, from equation (ii), $y - x = 1$
So, $y = 1+x$

So, for x = 0, 1, 2 and 3; y = 1, 2, 3 and 4. Plotting the points (0, 1), (1,2), (3, 4) and jointing them, we obtain graph of y – x = 1, which is CD in the following figure-5.1

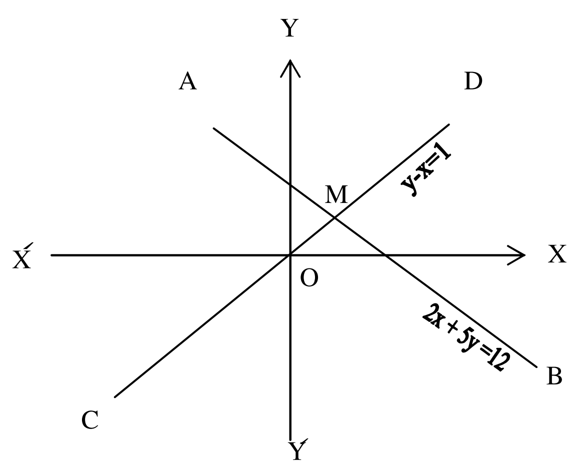


Figure-5.1

From the above graph it is clear that *AB* is the line of equation (i) and *CD* is the line of equation (ii) they are intersecting each other at the point *M*. The co-ordinates of *M* are (1, 2). Hence the required equation is $x = 1$ and $y = 2$.

This gives the solution of $x = 1$, $y = 2$ for the pair of simultaneous equations $2x + 5y = 12$ and $y - x = 1$.

Example – 2:

Draw the graph and solve the following equations; $4y - y + 11$ and $24x - 6y + 2 = 0$

Solution:

The given equations are $4x - y + 11 = 0$

or, $-y = - 11 - 4x$

or, $-y = - (11 + 4x)$

so, $y = 11 + 4x$ .... (i)

and $24x - 6y + 2 = 0$

or, $-6y = - 24x - 2$

or, $-6y = - (24x+2)$

or, y $\frac{24x+2}{6}$ ....(ii)

We put the values 0, 1, 2 and 3 to *x,* and find corresponding values of *y* then putting down the values in the following table:

x	0	1	2	3
y = 11 + 4x	11	15	19	23
y = (24x + 2)/6	0.33	4.67	8.67	12.67

The system of equations has no solution, because the two lines of the equations are parallel and distinct.

The system of equations has no solution, because the two lines of the equations are parallel and distinct. Hence the given equations are inconsistent.

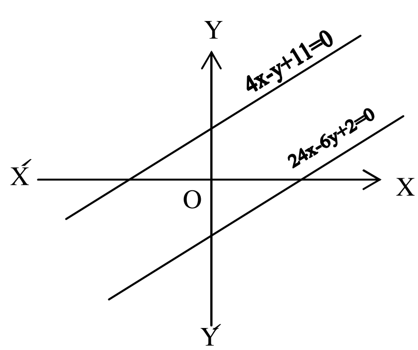


Figure-5.2

The system of equations has no solution, because the two lines of the equations are parallel and distinct. Hence the given equations are inconsistent.

Example–3:

Solve the following quadratic equation graphically;

$x^2 - 4x + 3 = 0$

Solution:

Let $y = x^2 - 4x + 3$

We give values 0, 1, 2, 3, 4 .... to x and find corresponding values of y put down in the following table:

x	0	1	2	3	4
x^2	0	1	4	9	16
$-4x$	0	–4	–8	–12	–16
+3	3	3	3	+3	3
y	3	0	–1	0	3

Plotting the points (0, 3), (1, 0), (2, –1), (3, 0), (4, 3) and jointing them, we get the graph $y = x^2 - 4x + 3$ on shown in the following figure 5.3.

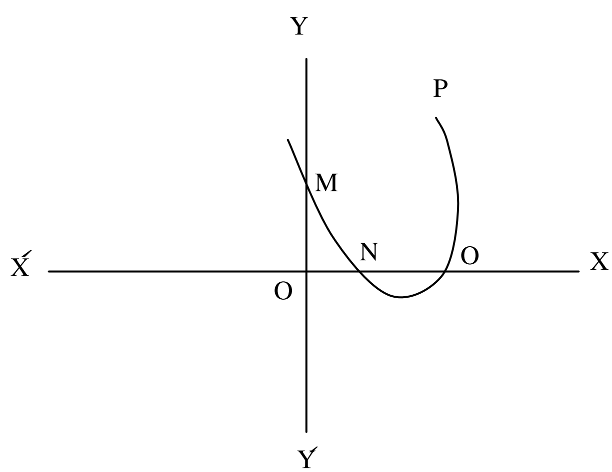


Figure-5.3

The curve MNOP is the graph of $x^2 - 4x + 3 = 0$. The graph of the function $y = x^2 - 4x + 3$ is the curve MNOP which crosses x axis at the points N and O where $x = 1$ and $x = 3$. Hence $x = 1$ and $x = 3$ are the solutions of the quadratic equation $x^2 - 4x + 3 = 0$.

Question for Review:

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. Draw the graphs and solve the following questions:

   (a) $12x - 7y = -6$
   $5x - 7y = 27$

   (b) $5x + 2y = 6$
   $y = 9x^2 + 16$

2. Draw the graph of the solution set for the system of liner inequalities $x \geq 0$
   $y \geq 0$
   $x + 2y \leq 45$
   $2x + y \leq 60$

3. Draw the graph of the solution set for the system of liner inequalities $2x + y \leq 30$
   $x - 2y \geq 20$
   $-4x + y \leq 39$

4. Draw the graphs and solve the equations
   (i) $x^2 - 5x + 6 = 0$
   $x + 2y + 7 = 0$

   (ii) $3x - 2y + 5 = 0$

Lesson–5: Quadratic Equation

After studying this lesson, you should be able to:

- State the nature of quadratic equation;
- Explain the relationship between roots and coefficient of quadratic equation;
- Explain the formation of quadratic equation; and
- Solve the quadratic equation.

Nature of Quadratic Equation

Generally an equation contains the square of unknown variable is called a quadratic equation. The general method of solving a quadratic equation of the form $ax^2 + bx + c = 0$ is given.

Generally an equation contains the square of unknown variable is called a quadratic equation.

Since $ax^2 + bx + c = 0$; by transposition $ax^2 + bx = c$. Hence dividing both sides by a, the co-efficient of x^2 we have,

$$x^2 + \frac{bx}{a} = -\frac{c}{a}$$

Adding to both sides $\left(\frac{b}{2a}\right)^2$

$$x^2 + \frac{bx}{a} + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$

$$\text{or, } x^2 + 2.x.\frac{b}{2a} + \left(\frac{b}{2a}\right)^2 = -\frac{b^2}{4a^2} - \frac{c}{a}$$

$$\text{or, } \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$\text{or, } x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

$$\text{or, } x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$\text{or, } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Thus the required roots of $ax^2 + bx + c = 0$ are $\frac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $\frac{-b - \sqrt{b^2 - 4ac}}{2a}$

Relationship between Roots and the Co-efficient of Quadratic Equation

A quadratic equation has exactly two roots.

A quadratic equation has thus exactly two roots. The relationship between the roots and the co-efficient of the quadratic equations as as follows:

Let the quadratic equation is $ax^2 + bx + c = 0$, then if α and β denote the roots of this quadratic equation, we have

$$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a} \qquad \beta = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

Therefore by addition,

$$\alpha + \beta = \frac{-b+\sqrt{b^2-4ac}}{2a} + \frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$\frac{-b+\sqrt{b^2-4ac}-b-\sqrt{b^2-4ac}}{2a}$$

$$\frac{-2b}{2a} = -\frac{b}{a}$$

And by multiplication,

$$\alpha\beta = \frac{-b+\sqrt{b^2-4ac}}{2a} \times \frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$\frac{(-b)^2 - \left(\sqrt{b^2-4ac}\right)^2}{4a^2}$$

$$\frac{b^2-b^2+4ac}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}$$

Thus, we have shown that

Sum of the roots $(\alpha+\beta) = -\frac{b}{a} = -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2}$

Product of the roots $(\alpha\beta) = \frac{c}{a} = \frac{\text{Constant term}}{\text{Coefficient of } x^2}$

Formation of Quadratic Equation

The formation of the quadratic equation whose roots are given can be explained as follows:

Let the general form of quadratic equation is $ax^2 + bx + c = 0$ and α and β denote the two given roots, where *a, b* and *c* are constants whose values we have to find out.

The sum of the roots $\alpha + \beta = -\frac{b}{a}$,

$$\text{So, } b = -a(\alpha+\beta)$$

And product of the roots, $\alpha\beta = \frac{c}{a}$

$$\text{so, } c = a\,\alpha\beta$$

The above relations imply that

$$ax^2 + bx + c = 0$$

$$\text{or, } ax^2+[-a(\alpha+\beta)]x + a\alpha\beta = 0$$

$$\text{or, } ax^2 - a\alpha x - a\beta x + a\alpha\beta = 0$$

$$\text{or, } x^2 - x(\alpha+\beta) + \alpha\beta = 0 \text{ [Dividing both sides by ‘a’]}$$

So, the required quadratic equation will be

x^2 − (sum of the roots)x + (product of the roots) = 0

The following section of this lesson contains some model applications of quadratic equation.

Example–1:

Solve $x^2 - 4x + 13 = 0$

Solution:

$x^2 - 4x + 13 = 0$

We know that $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Here a = 1, b = −4, c = 13

Substituting the given values we have

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4.(1)(13)}}{2.1}$$

$$= \frac{4 \pm \sqrt{16 - 52}}{2}$$

$$= \frac{4 \pm \sqrt{-36}}{2}$$

$$= \frac{4 \pm 6i}{2} \quad \left[\text{Since, } i = \sqrt{-1}\right]$$

Therefore, x = 2 + 3i or, 2 − 3i

Example–2:

Solve $3x + 2\sqrt{x} = \dfrac{10\sqrt{x}}{6x\sqrt{x} - 4x}$

Solution:

We have $3x + 2\sqrt{x} = \dfrac{10\sqrt{x}}{6x\sqrt{x - 4x}}$

$$or,\ 3x + 2\sqrt{x} = \frac{2\sqrt{x}.5}{2x\sqrt{(3x - 2\sqrt{x})}}$$

$$or,\ \left(3x + 2\sqrt{x}\right)\left(3x - 2\sqrt{x}\right) = 5$$

$$or,\ \left[(3x)^2 - 2\left(2\sqrt{x}\right)^2\right] = 5$$

or, $9x^2 - 4x - 5 = 0$

We know that x = $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Here a = 9; b = −4; c = −5

Substituting the given values we have

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(9)(-5)}}{2.9}$$

$$= \frac{4 \pm \sqrt{16 + 180}}{18}$$

$$= \frac{4 \pm \sqrt{196}}{18}$$

$$= \frac{4 \pm 14}{18}$$

$$So, x = \frac{18}{18} \quad or, \quad \frac{-10}{18}$$

Example–3:

If α and β are the roots of the equation $x^2 - px + q = 0$,

find the values of (i) $\alpha - \beta$ (ii) $\alpha^2 + \beta^2$ (iii) $\alpha^3 - \beta^3$.

Solution:

Since α and β are the roots of the equation $x^2 - px + q = 0$

So, sum of the roots, $\alpha + \beta = -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2} = -\frac{-P}{1} = P$

Product of the roots, $\alpha\beta = \frac{\text{Constant term}}{\text{Coefficient of } x^2} = \frac{q}{1} = q$

(i) $(\alpha - \beta)^2 = \alpha^2 + \beta^2 - 2\alpha\beta$

$= \alpha^2 + \beta^2 + 2\alpha\beta - 4\alpha\beta = (\alpha+\beta)^2 - 4\alpha\beta = p^2 - 4q$

so, $\alpha - \beta = \sqrt{p^2 - 4q}$

(ii) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = p^2 - 2q$

(iii) $\alpha^3 - \beta^3 = (\alpha + \beta)(\alpha^2 - \alpha\beta + \beta^2)$

$= (\alpha + \beta)[(\alpha - \beta)^2 + \alpha\beta]$

$= p(p^2 - 4q + q) = p(p^2 - 3q)$.

Example–4:

The roots of $x^2 - x + 1 = 0$ are α and β; from a quadratic equation whose roots are $\alpha^4 + \beta^4$ and $\alpha^2 + \beta^4$

Solution:

Since α and β are the roots of the equation $x^2 - x + 1 = 0$

Sum of the roots, $\alpha + \beta = -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^2} = -\frac{-1}{1} = 1$

And product of the roots, $\alpha\beta = \frac{\text{Constant term}}{\text{Coefficient of } x^2} = \frac{1}{1} = 1$

Now, the sum of the roots of the required equation,

$= \alpha^2 + \beta^2 + \alpha^2 + \beta^4$

$= (\alpha^2 + \beta^2) + [(\alpha^2 + \beta^2)^2 - 2\alpha^2\beta^2]$

$= [(\alpha+\beta)^2 - 2\alpha\beta] + [(\alpha+\beta)^2 - 2\alpha\beta]^2 - 2(\alpha\beta)^2$

$= (1^2 - 2.1) + [(1)^2 - 2.1]^2 - (1)^2$

$= - 1 + 1 - 2 = - 3 + 1 = - 2.$

And the product of the roots of the required equation.

$= (\alpha^4 + \beta^2)\ (\alpha^2 + \beta^4)$

$= \alpha^6 + \alpha^4\beta^4 + \alpha^2\beta^2 + \beta^6$

$= (\alpha^6+\beta^6)\ (\alpha\beta)^4 + (\alpha\beta)^2$

$= (\alpha^2+\beta^2)\ (\alpha^4 - \alpha^2\beta^2 + \beta^4) + (\alpha\beta)^4 + (\alpha\beta)^2$

$= [(\alpha+\beta)^2 - 2\alpha\beta]\ \ [(\alpha^2+\beta^2)^2 - 3\alpha^2\beta^2] + (\alpha\beta)^2 + (\alpha\beta)^2$

$= [(\alpha+\beta)^2 - 2\alpha\beta]\ \ [\{(\alpha+\beta)^2 - 2\alpha\beta\}^2 - 3(\alpha\beta)^2] + (\alpha\beta)^4 + (\alpha\beta)^2$

$= [(1)^2 - 2(1)]\ \ [\{(1)^2 - 2(1)\}^2 - 3(1)^2] + (1)^2 + (1)^{2}$

$= (- 1)\ (1 - 3) + 1 + 1 = 2 + 1 + 1 = 4.$

Hence the required quadratic equation is,

x^2 − (sum of the roots) x + (product of the roots) = 0

or, $x^2 - (-2)\ x + 4 = 0$

so, $x^2 + 2x + 4 = 0$

Questions for Review:

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. What is a quadratic equation? Give an example

2. State the characteristics of a quadratic equation.

3. If α and β are the roots of $x^2 - px + q = 0$, form a quadratic equation whose roots are $(\alpha\beta - \alpha - \beta)$.

4. If α and β are the roots of $2x^2 + 3x + 7 = 0$, find the values of (1) $\alpha^2 + \beta^2$ (ii) α/β (iii) β/α (iv) $\alpha^3 + \beta^3$

5. If the equation $ax^2 + bx + c = 0$ and $bx^2 + cx + a = 0$ have a common root, then prove that $a + b + c = 0$ and $a = b = c$.

6. Solve: (i) $3x^2 - 2x - 5 = 0$ (ii) $x^2 + 4x + 4 = y$

Multiple Choice Questions (√ the appropriate answers)

1. The roots of a quadratic equation are 5 and − 2. The equation is:

   (a) $x^2 - 3x - 10 = 0$ (b) $x^2 - 3x + 10 = 0$ (c) $x^2 + 3x - 10 = 0$

2. The quadratic equation whose one root is $3 + 2\sqrt{3}$, is

   (a) $x^2 + 6x - 10 = 0$ (b) $x^2 - 6x - 3 = 0$ (c) $x^2 + 6x + 3 = 0$

3. If α and β are the roots of the question $x^2 - 8x + P = 0$ and $\alpha^2+\beta^2 = 40$ then P is equal to:

   (a) 8 (b) 12 (c) − 12

4. If α and β are the roots of the equation $2x^2 - 4x + 3 = 0$ the value of the $\alpha^3 + \beta^3$ is:

   (a) − 1 (b) 5 (c) 2

5. If α and β are the roots of the equation $2x^2 - 3x + 1 = 0$, then the equation whose roots are α/β and β/α is:

   (a) $2x^2 + 5x + 2 = 0$ (b) $2x^2 - 5x + 2 = 0$ (c) $2x^2 +5x - 2 = 0$

6. If one root of the equation $3x^2 - 10x + 3 = 0$ is 1/3, then the other root is:

   (a) 3 (b) − 1/3 (c) −3

Lesson-6: Application of Equation in Business Problems

After studying this lesson, you should be able to:

- Apply the principles of equations to solve the business problems

Introduction

The application of equations in the solution of business problems is of great use.

The application of equations in the solution of business problems is of great use. It is usual to take the following steps in the solution of business problems.

(i) Read the problem very carefully and try to understand the relation between the quantities stated therein.

(ii) Denote the unknown quantity by the letter $x, y, z,$ etc.

(iii) Translate the statements of the problem step by step into mathematical statements; to the extent it is possible.

(iv) Solve the equation for the unknown

(v) check whether the obtained solution satisfies the condition given in the problem.

The following section of this lesson contains some model applications of equations to solve business problems.

Example-1:

A man says to his son, "Seven years ago I was seven times as old as you were, and three years hence, I shall be three times as old as you will be". Find their present ages.

Solution:

Let the present age of the son be x years.

His age seven years ago = $x - 7$

The father's age 7 years ago = $7(x - 7)$

Father's present age = $7(x - 7) + 7$.

Son's age 3 years hence = $x + 3$

Father's age 3 years hence = $7(x - 7) + 7 + 3 = 7(x - 7) + 10$

Using the given information we can write

$7(x - 7) + 10 = 3(x + 3)$

or, $7x - 49 + 10 = 3x + 9$

or, $7x - 3x = 49 + 9 - 10$

or, $4x = 48$

so, $x = 12$

So, son's present age = 12 years.

Father's present age = $7(12 - 7) + 7$

$= 35 + 7 = 42$ years.

Example-2:

A man's annual income has increased for Tk.7,500 but there is no change in the income tax payable for him since the rate of income tax has been reduced from 10% to 7%. Find his present income.

Solution:

Let previous annual income = x

$\therefore$Present income = x + 7500

His income tax of previous year = 10% of x = 0.10x

His income tax for present year = 70% of (x + 7500) = 0.07x + 525

According to the question we can write

$0.10x = 0.07x + 525$

or, $0.10x = 0.07x + 525$

or, $0.03x = 525$

or, $x = (525 \div 0.03)$

so, $x = 17{,}500$

So, the present income = (x + 7500) = (17,500 + 7500) = Tk.25,000.

Example-3:

Tk.20,000 is invested in two shares. The first yield is Tk.12% p.a. and the second yield is 14% p.a. If the total yield at the end of one year is 13% p.a.; how much was invested at each rate?

Solution:

Let x in Tk. be invested at 12%.

Then (Tk.20000 – x) is invested at 14%

Interest at 12% = 12% of x = 0.12x

Interest at 14% = 14% of (20,000 – x) = 2800 – 0.14x

Total Interest on 20,000 at 13% = 13% of 20,000 = 2,600

According to the given information we can write

$260 = 2800 - 0.14x + 0.12x$

or, $0.02x = 2800 - 2600$

or, $x = (200 \div 0.02) = 10{,}000$

Hence Tk.10,000 is invested at 12% and (20,000 – 10,000) = Tk.10,000 is invested at 14%.

Example-4:

Demand and supply equations are $2p^2 + q^2 = 11$ and $p + 2q = 7$ respectively. Find the equilibrium price and quantity. (Where p stands for price and for quantity).

Solution:

The demand function is : $2p^2 + q^2 = 11$ .... (1)

And supply function is : $p + 2q = 7$

So, $p = 7 - 2q$ .... (2)

Putting the value of p in equation (1) we have,

$2\,(7 - 2q)^2 + q^2 = 11$

or, $2\,(49 - 28q + 4q^2) + q^2 = 11$

or, $98 - 56q + 8q^2 + q^2 = 11$

or, $9q^2 - 56q + 98 - 11 = 0$

or, $9q^2 - 56q + 87 = 0$

or, $9q^2 - 27q - 29q + 87 = 0$

or, $9q\,(q - 3) - 29\,(q - 3) = 0$

or, $(q - 3)\,(9q - 29) = 0$

either, $q - 3 = 0$ or, $9q - 29$

so, $q = 3$ or, $q\ \frac{29}{9}$

When $q = 3$; the price is, $p = 7 - 2q = 7 - 2 \times 3 = 7 - 6 = 1$.

When $q = \frac{29}{9}$, the price is, $p = 7 - 2 \times \frac{29}{9} = 7 - \frac{58}{9} = \frac{63 - 58}{9} = \frac{5}{9}$

Therefore the equilibrium price and quantity are:

$(p, q) = (1, 3)$ or $\left(\frac{5}{9}, \frac{29}{9}\right)$

Questions for Review:

These questions are designed to help you assess how far you have understood and apply the learning you have accomplished by answering (in written form) the following questions:

1. Mr. Sajib invested Tk.16,000 in two types of debentures of Tk.100 each. 13% debentures were purchased at Tk.150 each and 10% debentures were purchased at Tk.120 each. If he got interest of Tk.970 after 1 year, find the sum invested in each type of debenture.
2. The ages (in years) of Ramesh and Rahim are in the ratio 5 : 7. If Ramesh were 9 years older and Rahim were 9 years younger, the age of Ramesh would have been were twice the age of Rahim. Find their ages.
3. If the total manufacturing cost '*y*' of making *x* units of a toy is: $y = \frac{25x - 9000}{2}$
   (i) What is the variable cost per unit?
   (ii) What is the fixed cost?
   (iii) What is the average cost of manufacturing 5,000 units?
   (iv) What is the marginal cost of producing 3,000 units?
   (v) What will be the effect on average cost per unit if volume changes?

Multiple choice questions (√ the appropriate answers)

1. The total cost of 6 books and 4 pencils is Tk.34 and that of 5 books and 5 pencils is Tk.30. The costs of each book and each pencil respectively are:

   a) Tk.5 and Tk.1 b) Tk.1 and Tk.5 c) Tk.6 and Tk.1

2. If 3 chairs and 2 tubes cost Tk.1200 and 5 chairs and 3 tubes cost Tk.1900, then the cost of 2 chairs and 2 tubes is:

   a) Tk.1000 b) Tk.900 c) Tk.700

3. The monthly income of A & B are in the ratio 4 : 3, Each of them saves Tk.600. If the ratio of the expenditures is 3 : 2, then the monthly income of A is:

   a) Tk.1800 b) Tk.2400 c) Tk.2000

4. If the laws of demand and supply are respectively given by the equation $4q + 9p = 48$ and $p = \frac{q}{9} + 2$, then the value of the equilibrium price and quantity respectively are:

   a) $^8/_3$ and 6 b) $^3/_8$ and 5 c) 8 and 6

Coordinate Geometry and the Straight Line

6

This unit aims to explain the concept of the coordinate geometry and the straight line. The unit are prepared with topics such as quadrants and coordinates of midpoints, distance between two points, the straight line, different forms of equation of a straight line and their application is solving business problems.

Blank Page

Lesson-1: Coordinate Geometry

After studying of this lesson, you should be able to:

- Explain the nature of coordinate geometry;
- Identify the quadrants;
- Identify the coordinates of any point;
- Calculate the coordinates of mid points;
- Calculate the distance between two points.

Nature of Coordinate Geometry

Coordinate geometry is that branch of geometry in which two real numbers, called coordinates, are used to indicate the position of a point in a plane. The main contribution of coordinate geometry is that it has enabled the integration of algebra and geometry. This is evident from the fact that algebraic methods are employed to represent and prove the fundamental properties of geometrical theorems. Equations are also employed to represent the various geometric figures.

Quadrants

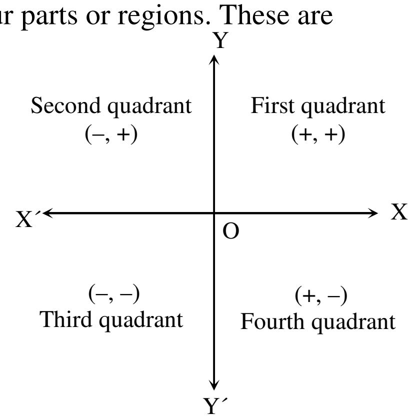
The two directed lines, when they intersect at right angles at the point of origin, divide their plane into four parts or regions. These are

XOY : First quadrant

X´OY : Second quadrant

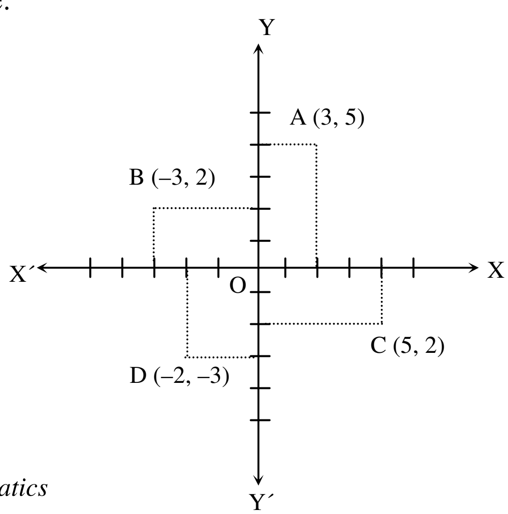
X´OY´ : Third quadrant.

XOY´ : Fourth quadrant



The position of the coordinates in a particular quadrant would depend on the positive and negative values of the

The position of the coordinates in a particular quadrant would depend on the positive and negative values of the coordinates shown in the following figure:



Coordinates

> *In a two-dimensional figure a point in plane has two coordinates.*

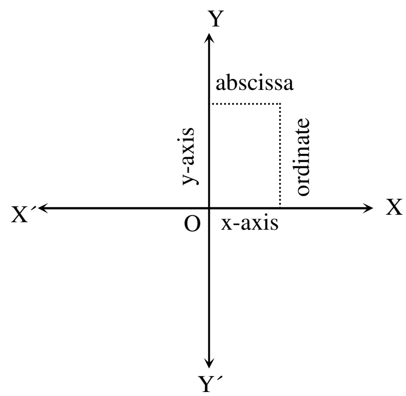
In a two-dimensional figure a point in plane has two coordinates. Generally, the first coordinate is read on the X´OX axis and the second coordinate on the Y´OY axis. Various methods of expressing these pairs of coordinates are:

(a) Varying alphabets *(x, y) (a, b)* etc.

(b) Varying subscripts (x_1, y_1) (x_2, y_2) etc.

(c) Varying dashes (x´´, y´´)

The diagrammatic presentation of the two coordinates is as follows:



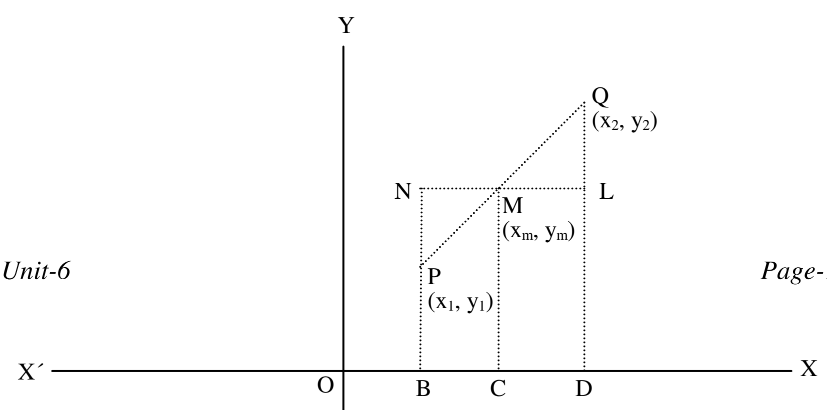
It is observed that the horizontal distance of the point from the Y´OY axis is called the x-coordinate or the abscissa and the vertical distance of the point from the X´OX axis is called the y-coordinate or the ordinate.

Coordinates of Mid-Points

We can find out the coordinates of a mid-point from the coordinates of any two points using the following formula:

$$X_m = \frac{x_1 + x_2}{2} \quad \text{and } Y_m = \frac{y_1 + y_2}{2}$$

This is helpful first in finding out the middle point from a join of any two points and secondly in verifying whether two straight lines bisect each other.



In the above figure, the dotted vertical lines are drawn perpendicular to x-axis and the dotted horizontal lines are parallel to the x-axis. The ΔNMP and ΔQML are the congruent triangles. Therefore NM = ML.

Accordingly BC = CD

or, OC − OB = OD − OC

or, $(x_m - x_1) = (x_2 - x_m)$

$\therefore x_m = \frac{x_1 + x_2}{2}$ =........ (1)

Also from the same congruent triangles we set

NP = QL

or, NB − PB = QD − LD

or, MC − PB = QD − MC

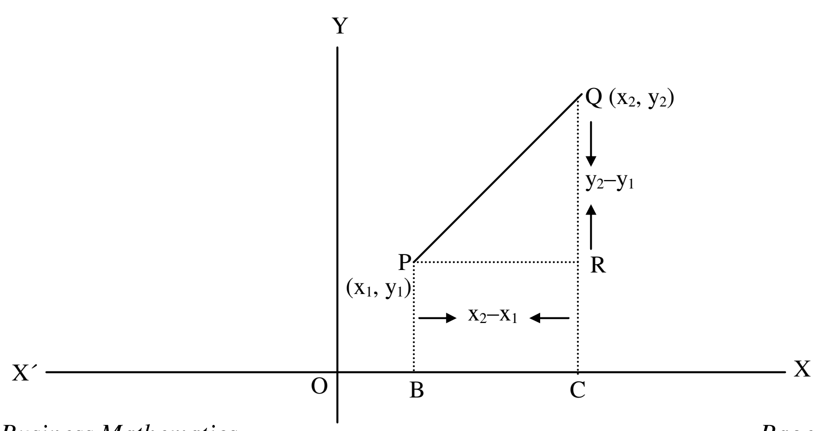
or, $y_m - y_1 = y_2 - y_m$

$\therefore y_m = \frac{y_1 + y_2}{2}$ =........ (2)

From (1) and (2), we conclude that the coordinates of the mid-point (x_m, y_m) are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

Distance between Two Points

Consider any two points P and Q with coordinates (x_1, y_1) and (x_2, y_2) respectively. By completing the right-angled triangle PRQ, we have the coordinates of R as (x_2, y_1)



Hence PR = $(x_2 - x_1)$ and QR = $(y_2 - y_1)$

Using Pythagoras theorem,

$PQ^2 = PR^2 + QR^2$

$= (x_2 - x_1)^2 + (y_2 - y_1)^2$

$\Rightarrow PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)}$

The general formula for the distance between any two given points P(x_1, y_1) and Q (x_2, y_2) is

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{(difference\ of\ abscissa)^2 + (difference\ of\ ordinates)^2}$$

Section Formula

The coordinates of a point R (x, y) dividing a line in the ratio of m : n connecting the points P (x_1, y_1) and Q (x_2, y_2). The coordinates of the point R using the following formula:

$$x = \frac{mx_2 + n\ x_1}{m+n} \text{ and } y = \frac{my_2 + n\ y_1}{m+n}$$

The following examples illustrate the model applications of coordinate geometry.

Example-1:

Find the coordinates of the mid-point of the straight line joining P (5, −4) and Q (−1,10)

Solution:

We know that the coordinates of the mid-point of the line joining two of points (x_1, y_1) and (x_2, y_2) are: $X_m = \frac{x_1 + x_2}{2}$ and $Y_m = \frac{y_1 + y_2}{2}$

$\therefore$ The coordinates of the mid-point of the straight line joining P (5, −4) and Q (−1, 10) are

$$X_m = \frac{5-1}{2} = 2 \text{ and } Y = \frac{-4+10}{2} = 3$$

i.e. the midpoint is M (2, 3)

Example-2:

Find the distance between the points (−2, 3) and (1, −3)

Solution:

Let (−2, 3) be denoted by (x_1, y_1) and (1, −3) be denoted by (x_2, y_2)

Therefore the required distance is, $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$$= \sqrt{[1-(-2)]^2 + (-3-3)^2}$$

$$= \sqrt{9+36} = \sqrt{45} = 3\sqrt{5} \text{ units.}$$

Example-3:

Show that the points (6, 6) (2, 3) and (4, 7) are the vertices of a right-angled triangle.

Solution:

Let P, Q, R be the points (6, 6), (2, 3) and (4, 7) respectively, then

$$PQ^2 = [(6-2)^2 + (6-3)^2] = 16 + 9 = 25$$

$$QR^2 = [(2-4)^2 + (3-7)^2] = 4 + 16 = 20$$

$$RP^2 = [(4-6)^2 + (7-6)^2] = 4 + 1 = 5$$

$$\therefore PQ^2 = QR^2 + RP^2$$

$$\Rightarrow \angle PQR = 1 \text{ right angle.}$$

Hence the points P(6, 6), Q (2, 3) and R (4, 7) are the vertices of a right angled triangle.

Example – 4:

Prove that (−2, −1), (1, 0), (4, 3) and (1, 2) be the vertices of a parallelogram.

Solution:

Let P (−2, −1), Q (1, 0), R (4, 3) and S (1, 2) be the vertices of a quadrilateral.

Then the midpoint of PR = $\left(\frac{-2+4}{2}, \frac{-1+3}{2}\right)$ = (1, 1) ... (i)

and the midpoint of QS = $\left(\frac{1+1}{2}, \frac{0+2}{2}\right)$ = (1, 1) ... (ii)

From (i) and (ii), we conclude that PR and QS bisect each other at the same point (1,1) and hence the quadrilateral PQRS is a parallelogram.

Example – 5:

Determine the coordinates of the vertices of the triangle PQR if the middle points of its sides PQ, QR and RP have coordinates (2, 5), (–4, 3) and (4, –1) respectively.

Solution:

Let the coordinates of the point P, Q and R of the triangle PQR are (x_1, y_1), (x_2, y_2) and (x_3, y_3) respectively. Therefore, we have:

For coordinates of the

midpoint of PQ : = $\frac{x_1 + x_2}{2} = 2;$ $\Rightarrow x_1 + x_2 = 4$ ... (i)

$\frac{y_1 + y_2}{2} = 5;$ $\Rightarrow y_1 + y_2 = 10$ ... (ii)

For coordinates of the

midpoint of QR : $\frac{x_2 + x_3}{2} = -4$ $\Rightarrow x_2 + x_3 = -8$ ... (iii)

$\frac{y_2 + y_3}{2} = 3;$ $\Rightarrow y_2 + y_3 = 6$ ... (iv)

For coordinates of

the midpoint of PR : $\frac{x_3 + x_1}{2} = 4;$ $\Rightarrow x_3 + x_1 = 8$ ... (v)

$\frac{y_3 + y_1}{2} = -1;$ $\Rightarrow y_3 + y_1 = -2$ ... (vi)

Adding (i), (iii) and (v), we have $x_1+x_2+x_2+x_3+x_3+x_1 = 4 - 8 + 8$

$\Rightarrow 2(x_1 + x_2 + x_3) = 4$

$\Rightarrow x_1 + x_2 + x_3 = 2$ ...(vii)

Adding (ii), (iv) and (vi), we have $y_1+y_2+y_2+y_3+y_3+y_1 = 10 + 6 - 2$

$\Rightarrow 2\ (y_1+y_2+y_3) = 14$

$\therefore\ y_1+y_2+y_3 = 7$ ... (viii)

Now, by (iii) and (vii), we have $x_1 = 10$

by (v) and (vii), we have $x_2 = -6$

by (i) and (vii), we have $x_3 = -2$

Again, by (iv) & (viii), we have $y_1 = 1$,

by (vi) & (viii), we have $y_2 = 9$

by (ii) & (viii), we have $y_3 = -3$

Therefore, the coordinates of the vertices of the triangle PQR are: P (10, 1), Q (−6, 9) and R (−2, −3)

Example – 6:

Find the co-ordinates of a point C dividing a line in the ratio of 7:3 connecting the points P (−2, 9) and Q (8, −1)

Solution:

Let the coordinates of the point C are x and y. Now by the formula

$$x = \frac{m\ x_2 + n\ x_1}{m + n} \text{ and } y = \frac{m\ y_2 + n\ y_1}{m + n}$$

we have, m = 7, n = 3, $x_1 = -2$, $x_2 = 8$, $y_1=9$ and $y_2 = -1$

$$\text{Hence } x = \frac{7x\ 8 + 3x\ (-2)}{7+3} = \frac{50}{10} = 5$$

$$\text{and } y = \frac{7x\ (-1) + 3x\ 9}{7+3} = \frac{20}{10} = 2$$

Therefore coordinates of the point C are (5, 2).

Questions for Review

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. Plot the points with the following coordinates
   P (−5, -4), Q (−4, 5), R (2, 4), S 1, −5)

2. Prove that the points (6.6), (2.3) and (4, 7) are the vertices of a right-angled triangle.

3. Determine the coordinates of the vertices of the triangle ABC if the middle points of its sides BC, CA, AB have coordinates (3, 2) (−1, −2) nad (5, −4) respectively.

4. If (−3, 2), (1, −2) and (5, 6) are the midpoints of the sides of a triangle, find the coordinates of the vertices of the triangle.

5. The points (3, 4), and (−2, 3) form with another point (x, y) an equilateral triangle. Find x and y.

6. Prove that the triangle with vertices at the points (0, 3) (−2, 1) and (−1, 4) is right angled.

Multiple Choice Questions (√ the most appropriate answer)

1. The distance between the points (2, −3) and (2, 2) is

   (a) 2 unit (b) 3 Unit (c) 4 Unit (d) 5 Unit

2. In which quadrant does (−4, 3) lie?

   (a) First quadrant (b) Second quadrant

   (c) Third quadrant (d) Fourth quadrant

3. The coordinates of a point situated on x-axis at a distance of 3 unit from y-axis is :

   (a) 0, 3) (b) 3, 0) (c) (3, 3) (d) (-3, 3)

4. The coordinates of a point below x-axis at a distance of 4 units from x-axis but lying on y-axis is

   (a) (0, 4), (b) (−4, 0) (c) (0, −4) (d) (4, −4)

5. The points A (0, 6), B (−5, 3), and C (3,1) are the vertices of a triangle which is

   (a) Isosceles (b) Right angled (c) Equilateral (d) None of these.

6. The area of a the triangle whose vertices are (3, 8) B (−4, 2) and C (5, −1) (in square units) is

   (a) 28 .5   (b) 37.5   (c) 75   (d) 57.

7. Which point of x-axis is equidistant from the points A (7, 6) and B (−3, 4)?

   (a) (0,4),   (b) (3, 0),   (c) (−4, 0),   (d) (0, 3)

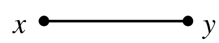
Lesson-2: The Straight Line

After studying this lesson, you should be able to:

- Discuss the nature of straight line;
- State the slope of a straight line;
- Highlights the different forms of equations of the straight line.

The Straight Line

If two points are given and if we join the points by a scale we get a straight line. Thus if x and y are two given points



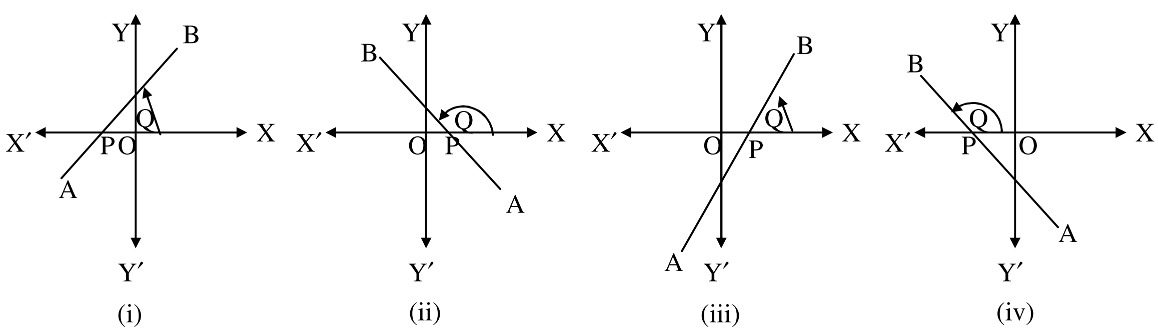
then *xy* is a straight line.

Straight line is the shortest distance between two distinct points.

Mathematically it is defined as the shortest distance between two distinct points. The study of curves starts with the straight line which is the simplest geometrical entity. Each point of a straight line have a slope. Hence let us discuss the slope of a straight line.

Slope of a Straight Line

The slope of the line is the tangent of the angle formed by the line above the x-axis towards its positive direction whatever be the position of the line as shown below:



Slope of a line is generally denoted by M. Thus if a line makes an angle Q with the positive direction of the x-axis, its slope is, M = tanQ.

If Q is acute (i & iii), slope is positive and if Q is obtuse (ii and iv), the slope is negative.

In terms of the co-ordinates, the slope of the line joining two points, say A (x_1, y_1) and B (x_2, y_2) is given by.

$$M = \tan Q = \frac{y_2 - y_1}{x_2 - x_1} = \frac{Difference\ \ of\ ordinates}{Difference\ \ of\ abscissae}$$

Different Forms of Equations of the Straight Line

1. ***Equations of the coordinate axes:*** All points on the x-axis, the value of y ordinates is always zero. Therefore y = 0 is the equation of x-axis. On the other hand, all points on the y-axis, the value of x ordinates is always zero. Hence x = 0 is the equation of y-axis.

2. ***Equations of lines parallel to the coordinate axes:*** Let P (x, y) be any point on a line parallel to x-axis at a distance *b* from it. Hence the equation of this line is y = b. Similarly x = a is the equation of the line parallel to the y-axis and at a distance *a* from it.

3. ***Origin-slope form:*** If the equation of a line passing through the origin and having the slop *m*, then the required equation of this line is, y = mx.

4. ***Slope-intercept form:*** If the equation of the line has the slope m with an intercept *c* on y-axis, then the required equation of this line is, y = mx + c.

5. ***Two-intercept form:*** Let a straight line intersect the coordinate axes making intercepts of *a* and *b* on x-axis and y-axis respectively, then the required equation of the line is;

$$y = \frac{a}{b}x + b \Rightarrow \frac{x}{a} + \frac{y}{b} = 1$$

6. ***Slope-point form:*** The equation of a straight line having a slope *m* and passing through the point (x_1, y_1) is.

$$y - y_1 = m\,(x - x_1)$$

7. ***Two-point form:*** If a straight line is passing through two points (x_1, y_1) and (x_2, y_2), then the equation of the straight line is,

$$\frac{y - y_1}{x - x_1} = \frac{y_1 - y_2}{x_1 - x_2} \Rightarrow y - y_1 = \frac{y_1 - y_2}{x_1 - x_2}(x - x_1)$$

Let us take some worked out examples on straight line.

Example-1:

Find the slope of the line joining points (0, 0) and (2, 3)

Solution:

The required slope = $\frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - 0}{2 - 0} = \frac{3}{2}$

Example-2:

Find the slope of the line whose equation is 2x + 3y − 7 = 0

Solution:

The equation 2x + 3y −7 = 0 can be written as 3y = −2x + 7

$\therefore\ y = -\frac{2}{3}x + \frac{7}{3}$.

Hence required slope is $-\frac{2}{3}$

Example-3:

Find the intercepts that the straight line 3x – 2y – 6 = 0 makes on the coordinate axes.

Solution:

Equation of the given line is

$$3x - 2y - 6 = 0$$

$$\text{or, } 3x - 2y = 6$$

$$\text{or, } \frac{3x}{6} - \frac{2y}{6} = 1$$

$$\text{or, } \frac{x}{2} - \frac{y}{3} = 1$$

$$\text{or, } \frac{x}{2} + \frac{y}{-3} = 1$$

Hence the intercepts made on the axes are 2 and −3.

Example-6.4:

What is the intercept that the straight line x – 2y + 4 = 0 makes on y-axis?

Solution:

The equation of the given line is

$x - 2y + 4 = 0$

or, $2y = x + 4$

or, $y = \frac{1}{2}x + 2$

Therefore the required intercept on y-axis is 2.

Questions for Review

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. Find the slope of the straight line joining the points (2, 5) and (−1, 3)
2. Find the intercepts that the line 3x – 4y + 12 = 0 makes on the axes. What is the slope of this line?
3. Find the slope of the line passing through the point (2, –1) and the origin.
4. Find the equation of the line through (1, –3) which is perpendicular to the line x – 3y + 4 = 0
5. Find the equation of the line making intercepts 2 and –3 on the x-axis and y-axis respectively.
6. Find the equation of the line through the origin and the point (–2, –6). What is the slope of the line?

Multiple Choice Questions (√ the appropriate answer)

1. The slope of the line joining A (−3, 5) and B (4, 2) is:

   (i) 3/7 (ii) 7/3 (iii) −3/7 (iv) −7/3

2. The equation of a line parallel to y-axis at a distance of 4 units to the right of y-axis is:

   (i) x = 4 (ii) y = 4 (iii) x = 4y (iv) y = 4x

3. he slope of the line 2x + 3y + 5 = 0 is:

   (i) 2 (ii) 3/2 (iii) -2/3 (iv) 3

4. he equation of a line with slope 4 and passing through the point, (5 −7) is:

   (i) y = 4x − 35 (ii) 4y = x − 35 (iii) y = 4x − 27 (iv) y = 4x + 27

5. The equation of the line passing through the point (1, 1) and perpendicular to the line 3x + 4y −5 = 0, is:

(i) $3x + 4y - 7 = 0$ (ii) $3x + 4y + k = 0$

(iii) $4x - 3y + 1 = 0$ (iv) $4x - 3y - 1 = 0$

6. The equation of a line passing through the origin and parallel to the line $3x - 2y + 1 = 0$, is:

   (i) $3x - 2y = 0$ (ii) $2x - 3y = 0$

   (iii) $2x - 3y - 1 = 0$ (iv) None

7. The equation of a line passing through (5, 1) and parallel to the line $7x - 2y + 5 = 0$, is:

   (i) $2x - 7y + 33 = 0$ (ii) $7x - 2y + 33 = 0$

   (iii) $2x - 7y - 33 = 0$ (iv) None.

8. The equation of a line parallel to y-axis and passing through (3, −7) is:

   (i) $x = 3$ (ii) $y = -7$ (iii) $y = 7x$ (iv) $y = 3x$

9. The length of perpendicular from the point (4, 1) to the line $3x - 4y + 12 = 0$, is:

   (i) 4 units (ii) 4/3 units (iii) 3 units (iv) 1 units.

10. The length of perpendicular from the origin to the line $3x + 4y + 5 = 0$, is:

    (i) 5 units (ii) 4/3 units (iii) 5/3 units (iv) 1 units.

Lesson-3: General Form of the Equation of a Straight Line

After studying of this lesson, you should be able to:

- ➢ State the nature of the general equation of a straight line;
- ➢ Highlights on some model applications of the problems related to straight line.

General Form of the Equation of a Straight Line

An equation of the form ax + by + c = 0, is called the general equation of the straight line

An equation of the form ax + by + c = 0, where a, b, c are constants and x, y are variables, is called the general equation of the straight line. Thus 2x + 3y + 7 = 0, 5x – y + 1 = 0, 3x + 2y = 0 are the equations of different straight lines in general form.

The equation ax + by + c = 0 can be written in slope intercept form as

by = –ax – c

or, $y = \frac{a}{b}x - \frac{c}{b}$, where $b \neq 0$

or, y = mx + k, where $m = -\frac{a}{b}$ and $k = -\frac{c}{b}$

i.e. slope of the line = $m = -\frac{a}{b} = \frac{-\text{coefficient of } x}{\text{coefficient of } y}$

Here *m* is the slope of the line whose equation is *y = mx + k,* where *k* is the intercept of the line on y-axis.

The equation *ax + by + c = 0* can also be written in two intercept from as follows:

ax + by = – c

or, $\frac{ax}{-c} + \frac{by}{-c} = 1,\ c \neq 0$

or, $\frac{x}{\frac{-c}{a}} + \frac{y}{\frac{-c}{b}} = 1$

or, $\frac{x}{a} + \frac{y}{b} = 1$ where $A = -\frac{c}{a}$ and $B = -\frac{c}{b}$

Here A and B are known as intercepts on the x-axis and y-axis respectively.

If a line passes through a point (x_1, y_1), its equation is $y - y_1 = m(x - x_1)$, where m is the slope of the line.

If a straight line passes through two points (x_1, y_1) and (x_2, y_2), its equation is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

The slope of this line is $\frac{y_2 - y_1}{x_2 - x_1}$

$$= \frac{\text{difference of } y \text{ - coordinates of the points}}{\text{difference of } x \text{ - coordinates of the points}}$$

When two points are given, we can find the slope of the line joining the points.

Thus when two points are given, we can find the slope of the line joining the points. For example, the slope of the line joining the points A (2, 3) and B (5, 4) is $\frac{4-3}{5-2} = \frac{1}{3}$

The following examples illustrate the coordinate geometry and the uses of straight line.

Example-1:

Find the equation of the straight line passing through the points Q (1, 2) and R (3, 7)

Solution:

Let a point P (x, y) lie on the same straight line.

Gradient of PQ = gradient of QR

$$\frac{y-2}{x-1} = \frac{7-2}{3-1}$$

or, $\frac{y-2}{x-1} = \frac{5}{2}$

or, 2y – 4 = 5x – 5

or, 2y = 5x – 1

or, $y = \frac{5x-1}{2}$

Therefore the equation of the straight line is, $y = \frac{5x}{2} - \frac{1}{2}$

Example-2:

A printer quotes a price of Tk.7,500 for printing 1,000 copies of a book and Tk.15,000 for printing 2,500 copies. Assuming a linear relationship and that 2,000 books are printed. Required:

(a) Find the equation relating to the total cost (y) and the number of books (x) printed.

(b) What is the variable cost of printing 2000 books?

(c) What is the fixed cost?

(d) What is the variable cost per book?

(e) What is the average cost per book to print 2000 books?

(f) What is the marginal cost of the last book printed?

Solution:

(a) Let x coordinate represents number of books and y coordinate represents the cost of printing.

Then the linear relationship between the number of books and cost of printing is the equation of the straight line passing through the points (1,000, 7,500) and (2,500, 15,000).

The equation of a straight line passing thought the points (x_1, y_1) and (x_2, y_2) is given by

$$y - y_1 = \frac{y_1 - y_2}{x_1 - x_2}(x - x_1) \qquad \text{... (1)}$$

Substituting the values of (x_1, y_1) and (x_2, y_2) in equation (1) we get

$$y - 75000 = \frac{75000 - 15000}{1000 - 2500}(x - 1{,}000)$$

$$\text{or, } y - 7{,}500 = \frac{-7500 - 15000}{-1500 - 2500}(x - 1000)$$

or, $y - 7500 = 59x - 1{,}000)$

or, $y - 7500 = 5x - 5000$

or, $y = 5x - 5000 + 7500$

or, $y = 5x + 2500$ ... (2)

The equation no. (2) represents the linear relationship between the number of books printed and cost of printing.

When 2000 books are printed, the cost of printing is given by, y = 5x + 2500.

= 5 (2000) + 2500 = 10,000 + 2500 = Tk.12,500

⇒ Total cost of printing 2000 books is Tk.12,500

(a) Comparing equation no. (2) with the slope-intercept form (y = m x +

(b) we have the variable cost given by, = (5 × 2000) = Tk.10,000

(c) The fixed cost is, C = 2500

(d) The variable cost per book is = $\frac{10000}{2000}$ = Tk.5

(e) Average cost per book is = $\frac{12500}{2000}$ = Tk.6.25

(f) Marginal cost = (Total cost of 2000 books – Total cost of 1999 books)

= 5(2000) + 2500 – [5(1999) + 2500]

= (12500 – 12495) = Tk.5

Example-3:

If the total manufacturing cost (y) of producing x units of a product is Tk.5,000 at 200 units output and Tk.7,250 at 300 units output and the cost-output relation is linear, then

(a) What is the equation of cost-output relationship in general form?

(b) What is the slope of the cost-output line?

(c) How much does the production of one unit add to total cost?

Solution:

We are given, the cost-output relation is linear and the information given consists of 2 points whose coordinates (x, y) are in the order (units made, total cost). These two points are: (200, 5000) and (300, 7250).

We know that the two-point form of the equation of a straight line is:

$$y - y1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$\Rightarrow\ y - 5000 = \frac{7250 - 5000}{300 - 200}(x - 200)$$

$$\Rightarrow\ y - 5000 = \frac{45}{2}(x - 200)$$

$$\Rightarrow 2y - 10{,}000 = 45x - 9{,}000$$

$$\Rightarrow\ -45x + 2y - 1{,}000 = 0$$

(a) Therefore, the equation of the cost-output relationship in general form is:
$-45x + 2y - 1000 = 0$

(b) The slope of the cost-output line is, $m = -\frac{\text{Coefficient of } x}{\text{Coefficient of } y}$

$$= -\frac{(-45)}{2} \qquad = \frac{45}{2} = 22.50$$

(c) Cost (y) of producing x units can be calculated by the cost-output equation as follows: $-45x + 2x - 1000 = 0$

$\Rightarrow 2y = 45x + 1000$

$\Rightarrow y = \frac{45}{2}x + 500$

And, the cost of producing (x+1) units is, $y = \frac{45}{2}(x+1) + 500$

$$\Rightarrow y = \frac{45}{2}x + \frac{1045}{2}$$

$\Rightarrow$ The cost of production of one unit that adds to the total cost is:

$$\Rightarrow \left[\frac{45}{2}x + \frac{1045}{2}\right] - \left[\frac{45}{2}x + 500\right]$$

$$= \left(\frac{1045}{2} - 500\right) = \frac{45}{2} = \text{Tk.}22.5$$

Example-4:

The total expenses (y) of a mess are partly constant and partly proportional to the number of the inmates (x) of the mess. The total expenses are Tk.1040 when there are 12 members in the mess, and Tk.1600 for 20 members.

(i) Find the linear relationship between y and x

(ii) Find the constant expenses and the variable expenses per member, and

(iii) What would be the total expenditure if the mess has 15 members?

Solution:

Let x coordinate represents the number of inmates of the mess and y coordinate represents the total expenses.

The given two points are (12, 1040) and (20, 1600). The equation of the straight line passing through the points (x_1, y_1) and (x_2, y_2) is given by

$$y - y_1 = \frac{y_1 - y_2}{x_1 - x_2}(x - x_1)$$

$$\Rightarrow y - 1040 = \frac{1040 - 1000}{12 - 20}(x - 12)$$

$$\Rightarrow y - 1040 = \frac{-560}{-8}(x - 12)$$

$\Rightarrow$ y − 1040 = 70 (x - 12)

$\Rightarrow$ y = 70x – 840 + 1040 = 70x + 200 ... (1)

$\Rightarrow$ y = 70x + 200 which is the required relationship between x and y.

(ii) Comparing the equation (i) with slope-intercept form (y = mx + c) we find, the constant expenses (c) = Tk.200 and variable expenses per member (m) = 70

(iii) When the number of members in the mess is 15, the total expenses, y = (70 x 15 + 2000) = Tk.1250.

Example-5:

A firm invested Tk.10 Million in a new factory that has a net return of Tk.5,00,000 per year. An investment of Tk.20 million would yield a net income of Tk.2 million per year. What is the linear relationship between investment and annual income? What would be the annual return on an investment of Tk.15 million.

Solution:

Let x coordinate represents the investment and y coordinate represents the annual income. As the relationship between x and y is linear, we have to find the equation of line through (1,00,00,000; 500000) and (2,00,00,000; 20,00,000)

$\therefore$ The required relationship is $y - y_1 = \dfrac{y_1 - y_2}{x_1 - x_2}(x - x_1)$

$$\text{or,} \quad y - 5{,}00{,}000 = \frac{500000 - 2000000}{10000000 - 20000000}(x - 1{,}00{,}00{,}000)$$

$$\text{or,} \quad y - 5{,}00{,}000 = \frac{-1500000}{-10000000}(x - 1{,}00{,}00{,}000)$$

or, 20y − 1,00,00,000 = 3x − 300,00,000

or, 3x − 20y − 20000000 = 0

Again when investment x = 150,00,000, the annual income (y) can be found by putting the value of x in the equation obtained; i.e. 3 (150,00,000) – 20y – 20000000 = 0

or, –20y = –450,00,000 + 200,00,000
or, –20y = –250,00,000
or, y = 12,50,000.

Questions for Review

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. The total cost of y for x units of a certain product consists of fixed cost and the variable cost. It is know that the total cost is Tk.1,200 for 100 units and Tk.2,700 for 400 units.
   (i) Find the linear relationship between x and y.
   (ii) Find the slope of the line and what does it indicate?
   (iii) If the selling price is Tk.7 per unit, find the number of units that must be produced so that there will be neither profit nor a loss.

2. Find the equation of a straight line passing through the point of intersection of the lines x − 2y + 3 = 0, 2x − 3y + 4 = 0 and parallel to the line joining the points (1,1) and (0, −1).

3. The total cost y, for x units of a certain product, consists of fixed cost and variable cost (proportional to the number of units produced). It is known that the total cost is Tk.6000 for 500 units and Tk.9000 for 1000 units.

   Based on the above information, you are required to find out.
   (i) The linear relationship between x and y;
   (ii) The slope of the line, what does it indicate?

(iii) The number of units that must be produced so that

   (a) There is neither profit nor a loss

   (b) There is a profit of Tk.1,000.

   (c) There is a loss of Tk.300; it is given that the selling price is Tk.8 per unit.

4. A firm invests Tk.10,000 in a business which has a net return of Tk.500 per year. An investment of Tk.20,000 would yield an income of Tk.2,000 per year. What is the linear relationship between investment and annual income? What would be the annual return on an investment of Tk.12,000?

5. If the total manufacturing cost (y) of producing x units of a product is Tk.5,000 at 200 units output and the Tk.7,250 at 300 units output and the cost-output relation is linear, then

   (a) What is the equation of cost- output relationship in general form?

   (b) What is the slope of the cost-output line?

   (c) How much does the production of one unit add to the total cost?

Functions, Limit and Continuity of a Function

7

From the discussion of this unit, students will be familiar with different functions, limit and continuity of a function. The principal foci of this unit are nature of function and its classification, some important limits and continuity of a function and its applications followed by some examples.

Blank Page

Lesson-1: Functions

After studying this lesson, you should be able to:

- Discuss the nature of variable and constants;
- State the functions and its classification;
- Highlights on some worked out examples related to the functions.

Introduction:

First of all we have to know some important terms, which are frequently used in this lesson. These are:

- **Constant**

  A constant is a symbol - which never changes over the sets of mathematical operation. For example, 1, 2, 3 are constants. The letter *a, b, c* --- are also considered as constants which are specially know as arbitrary constants.

- **Variables**

  A symbol capable of assuming different values is called a variable. Variable are usually denoted by the letters of the alphabet; i.e., *x, y, z*.

- **Independent Variable**

  A variable to which any value can be assigned is called an independent variable. Independent variables are the causes and the dependent variables are the effects.

- **Dependent Variable**

  A variable whose value depends on the value of the independent variable is called a dependent variable.

- **Function**

  When two variables are so related that one is dependent and another is independent, then the dependent variable is known as function of independent variable. For example, let us consider two variables x and y, which are related by the equation $y = 4x + 6$. We see that if we take $x = 1$, then we get $y = 10$; if we take $x = 0$, we get $y = 6$ and thus we see here that x is independent variable and y is dependent variable. So we may say that y is the function of x which is denoted by the symbol, $y = f(x)$. Hence we may conclude that any expression containing a variable is called a function of that variable. Thus *(i)* $ax + 10$, *(ii)* $2x^2 - 5x + 2$, *(iii)* $t^2 - 1$ and *(iv)* $e^t - 5$, where the expressions *(i)* and *(ii)* are functions of x and the expressions *(iii)* and *(iv)* are functions of t. The related variable on which the value of the function depends is also known as argument of the function.

When two variables are so related that one is dependent and another is independent, then the dependent variable is known as function of independent variable

Type of Functions

The different types of functions have been discussed as under:

a) **Linear Function:** A linear function represents a relationship between two variables, i.e., one dependent variable and another independent variable. Generally the functional form of the linear function is, $f(x) = ax + b$

where, $f(x)$ is the dependent variable
x is the independent variable
b is the value of the dependent variable when x is zero.
a is the coefficient of the independent variable.

The above symbol $f(x)$ is read as *"function of x"*, which represents the values of the dependent variable, and x represents values of the independent variable; $f(x)$ varies according to the rule of the function as x varies. For a linear function, the rule of the function states that *'a'* is to be multiplied by x and this product is to be added to b. This sum determines the value of the dependent variable $f(x)$.

b) **Quadratic Function:** The quadratic function is a second degree function which has important applications in business and economics. The general form of the quadratic function is, $f(x) = ax^2 + bx + c$

where, $f(x)$ is the dependent variable
x is the independent variable
a, b and c are the parameters of the functions.

The shape of the quadratic function is determined by the magnitude and signs of the parameters a, b and c.

c) **Polynomial Functions:** Linear and quadratic functions belong to the class of functions termed polynomial functions. The general form of the polynomial function is

$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + ... + a_nx^n$

where, $a_0, a_1, a_2, a_3, ... a_n$ are parameters and n is a positive integer.

The parameters may be positive, negative or zero. The polynomial function is linear if $n=1$ and quadratic if $n=2$. This can be verified by comparing this for $n=1$ with the general form of the linear function, and $n=2$ with the general from of the quadratic function.

A polynomial function in which the largest exponent is n=3 is termed as cubic function

A polynomial function in which the largest exponent is $n=3$ is termed as cubic function. The general form of the cubic function is $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3$

d) **Multivariate Functions:** Functions in which the single dependent variable is related to more than one independent variable are termed as multivariate functions. The general form of multivariate function is, $f(x_1, x_2) = 2x_1+5x_1 x_2+6x_2$
where $f(x_1, x_2)$ is the dependent variable
x_1 is an independent variable
and x_2 is a second independent variable.

e) **Exponential Functions:** The exponential function is a specific function in which a constant is raised to a variable power rather than a variable raised to constant power. This function with a variable power is called the exponential function. The general form of exponential function is , $h(x) = k.a^{f(x)}$

where *'a'* is a constant greater than zero and not equal to one and $f(x)$ is any real number function.

The domain of this function is the set of all real numbers, x, for which $f(x)$ is defined.

exponential tion states the stant rates of wth

The exponential function states the constant rates of growth. As the independent variable increases by a constant amount in the exponential function, the dependent variable increases or decreases by a constant percentage. Hence, the value of an investment that increases by a constant percentage each period, the sales of a company that increase at a constant rate each period, and the value of an asset that declines at a constant rate each period are examples of functional relationship that are described by the exponential functions.

f) **Logarithmic Functions:** The inverse of the exponential function is the logarithmic function. The general form of the logarithmic function is, $y = log_a x$

where, y is the dependent variable
x is the independent variable
and *'a'* is a constant, termed the base, that is greater than 0 and not equal to 1.

The logarithmic function arises when we ask the question, for what value of y is $a^y = x$.

If $a^y = x$, then $log_a x = y$ and vice-versa. Thus, the exponential function is corresponding to the logarithmic function, $y = log_a x$ is $a^y = x$.

Rate of Change

The rate of change of a function is the change in the value of the dependent variable with respect to the change in the value of the independent variable, i.e.,

$$\text{Rate of change} = \frac{\Delta f(x)}{\Delta x} = \frac{\text{vertical change}}{\text{horizontal change}} = \frac{\text{rise}}{\text{run}}$$

If the independent variable x increases by Δx, the new value of the independent variable is $(x + \Delta x)$. For a linear function when $f(x) = ax + b$, the new value of the dependent variable for change in x is $f(x + \Delta x) = a(x + \Delta x) + b$.

To determine the amount of change in the dependent variable as the independent variable changes by Δx, the old value of the dependent variable, $f(x)$ is subtracted from the new value of the dependent variable, $f(x + \Delta x)$. That is, for the linear case,

$$\Delta f(x) = f(x + \Delta x) - f(x) = a(x + \Delta x) + b - (ax + b)$$
$$= (ax + a\Delta x + b - ax - b) = a.\Delta x.$$

Hence for the linear case, the rate of change is $\dfrac{\Delta f(x)}{\Delta x} = \dfrac{a.\Delta x}{\Delta x} = a,$ which is the slope. The rate of change can be calculated for any function, linear or non-linear, using the same formula.

The rate of change can be calculated fo any function, linear non-linear, using th

Notations for Functions

It *'x'* is a variable of a function, then it is expressed as $f(x), F(x), g(x), \ldots f_1(x), f_2(x) \ldots$ which are basically called *functions of x*. Similarly it may be expressed as *'the f function of x'*, *'the F function of x'* ... etc.

Again, if more than one variable (*x, y, z*) exist in a particular function, it can be expressed as $f(x, y), F(x, y, z)$. It is termed as *'the function of x and y'*, *'the F function of x, y, and z'* etc.

For example, If $f(x) = 2x^3 - 5x + 3$ and $F(x, y) = 3x^c + 5e^y - 3xy$,

then, $f(p) = 2p^3 - 5p + 3$ and $F(b, d) = 3b^c + 3e^d - 3bd.$

If the interval is $a \leq x \leq b$, then it is called closed domain in which the values of a

If the value of *'x'* exists between *a* and *b* then it is termed as domain or interval. If the interval is $a \leq x \leq b$, then it is called closed domain in which the values of *a* and *b* are included.

Again if the interval is $a < x < b$, then it is called open domain, where the mid values of a and b are

Again if the interval is $a < x < b$, then it is called open domain, where the mid values of *a* and *b* are included only. The samples of functions are presented as under:

$f(x) = 3x + 5$ → It is a linear function

$f(x) = 3x^2 - 3x + 8$ → It is a quadratic function

$f(x) = 4x^3 - 9x^2 + 3x - 6$ →It is a cubic function.

The following examples illustrate the use of functions

Example-1:

If $p(q) = q^2 - r^2 + 5$ and $h(r) = q^2 - r^2 + 5$; what is (i) $p(2)$ and (ii) $h(3)$?

Solution:

(a) We are given, $p(q) = q^2 - r^2 + 5$

$$\therefore\ p(2) = (2^2 - r^2 + 5) = 9 - r^2$$

(b) We are given, $h(r) = q^2 - r^2 + 5$

$$\therefore\ h(3) = (q^2 - 3^2 + 5) = q^2 - 4$$

Example-2:

Find $g(64)$, If $g(x) = \dfrac{x^{\frac{3}{2}}}{32} - 16x^{-\frac{1}{2}} + 2x^{\frac{1}{3}}$

Solution:

We are given, $g(x) = \dfrac{x^{\frac{3}{2}}}{32} - 16x^{-\frac{1}{2}} + 2x^{\frac{1}{3}}$

$$\therefore\ g(64) = \left[\frac{64^{\frac{3}{2}}}{32} - 16(64)^{-\frac{1}{2}} + 2(64)^{\frac{1}{3}}\right] = (16 - 2 + 8) = 28$$

Example-3:

Find (i) $g(a) - g(x - a)$, if $g(x) = x^2 + 10$

(ii) $f(x + a) - f(x)$; if $f(x) = x^2 - 3$

Solution:

(i) We are given $g(x) = x^2 + 10$

$$\therefore\ g(a) - g(x - a) = (a^2 + 10) - \{(x - a)^2 + 10\}$$
$$= a^2 + 10 - x^2 + 2xa - a^2 - 10$$
$$= 2xa - x^2$$

(ii) We are given, $f(x) = x^2 - 3$

$$\therefore f(x + a) - f(x) = (x + a)^2 - 3 - (x^2 - 3)$$
$$= x^2 + 2xa + a^2 - 3 - x^2 + 3$$
$$= 2xa + a^2$$

Example-4:

If $f(x) = \frac{ax+b}{bx+a}$, prove that $f(x).\ f(\frac{1}{x}) = 1$

Solution:

We have $f(x) = \frac{ax+b}{bx+a}$

Replacing x by $\frac{1}{x}$ on both sides, we get

$$f(\frac{1}{x}) = \frac{a.\frac{1}{x}+b}{b.\frac{1}{x}+a} = \frac{a+bx}{b+ax}$$

$$\therefore f(x).f(\frac{1}{x}) = \frac{ax+b}{bx+a} \times \frac{a+bx}{b+ax} = 1 \text{ (Proved)}$$

Example-5:

Find the domain of the following function $\frac{x^2 + x + 5}{x^2 - 6x + 8}$

Solution:

Let $f(x) = \frac{x^2 + x + 5}{x^2 - 6x + 8}$

Clearly, $f(x)$ will be undefined if

$x^2 - 6x + 8 = 0$

or, $x^2 - 4x - 2x + 8 = 0$

or, $x(x-4) - 2(x-4) = 0$

or, $(x-4)(x-2) = 0$

$\therefore x = 2$ or $x = 4$

Hence, the domain of the definition of $f(x)$ is:

$-a < x < a$, but $x \neq 2$ and $x \neq 4$.

Example-6:

If $e^y - e^{-y} = 2x$, express y as an explicit function of x.

Solution:

We have $e^y - e^{-y} = 2x$ (Let $z = e^y$)

or, $z - \frac{1}{z} = 2x$

or, $z^2 - 1 = 2zx$

or, $z^2 - 2xz - 1 = 0$

$$\therefore z = \frac{2x \pm \sqrt{4x^2 - 4.1.(-1)}}{2.1}$$

$$\text{or, } e^y = \frac{2x \pm 2\sqrt{x^2+1}}{2}$$

or, $e^y = x \pm \sqrt{x^2+1}$

or, $\log_e e^y = \log_e (x \pm \sqrt{x^2+1})$

or, $y = \log_e (x \pm \sqrt{x^2+1})$, which expresses y as an explicit function of x.

Example-7:

The taxi fare is Tk.10 for 1 km or less from start and Tk.5 per km or any fraction thereof for additional distance. If the fare be Tk.y for a distance of x km, express y as a function of x.

Solution:

From the problem it is clear that

$y = 10$ when $0 < x \leq 1$

and $y = 10 + 5$ when $1 < x \leq 2$

$y = 10 + 2(5)$ when $2 < x \leq 3$

$y = 10 + 3(5)$ when $3 < x \leq 4$.

and in general,

$y = 10 + p(5)$ when $p < x \leq p+1$

when $p = 0$ or a positive integer.

Hence, the functional relation between y (in Tk.) and the distance traveled x (in km) is given by $y = 10 + p(5)$ when $p < x \leq p+1$, where $p = 0$ or a positive integer.

Example-8:

Find the range of the function $\frac{x}{1+x^2}$

Solution:

Let $y = \frac{x}{1+x^2}$

or, $x^2y + y = x$

or, $x^2y - x + y = 0$

$\therefore\ x = \frac{1+\sqrt{1-4y^2}}{2y}$

Since x is finite and real, we have

$y \neq 0$, and $1 - 4y^2 \geq 0$

or, $(1 - 2y)(1+2y) \geq 0$

$\therefore -1/2 \leq y \leq 1/2$

Therefore, the required range of the given function is: $-\frac{1}{2} \leq y \leq \frac{1}{2}$ *and* $y \neq 0$

or, $-1/2 \leq y < 0$ and $0 < y \leq 1/2$.

Questions for Review

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. What do you mean by constant and variable?
2. Define a function. What do you mean by domain interval and range of a function?
3. If $f(x - 1) = 7x - 5$, find $f(x)$ and $f(x+2)$
4. If $f(x) = x^2 - x$, then prove that $f(h+1) = f(-h)$
5. If $f(x) = \frac{1}{x^2}$ show that $f(x + h) - f(x - h) = -\frac{4xh}{(x^2-h^2)^2}$
6. If $f(x) = log_e \frac{1+x}{1-x}$, show that $f\left(\frac{2x}{1+x^2}\right) = 2f(x)$
7. If $y = f(x) = \frac{x-3}{2x+1}$ and $z = f(y)$, express z as a function of x.
8. Find the domain of the following function $\frac{x-2}{x^2-3x+2}$
9. Find the range of the function $\frac{x^2}{1+x^2}$

Multiple Choice Questions (Find the most appropriate answer)

1. Given $f(x) = 3x - 9$; find $f(x^2 - 1)$
   (i) $x^2 - 12$ (ii) $3x^2 - 12$
   (iii) $4x^2 - 6$ (iv) $3x^2 - 10$

2. Find the domain of the functions $y = \frac{x^2}{1+x^2}$
   (i) $x \le 2$ and $x \ge 5$ (ii) $x < -6$ and $x > e$
   (iii) $x > 2$ and $x < -1$ (iv) $a < x < a$

3. Find the range of the function $y = \frac{x}{x^2-5x+9}$
   (i) $0 \le y \le 2$ (ii) $0 \le y \le 1$
   (iii) $-\frac{1}{11} \le y \le 1$ (iv) $y \le \frac{1}{2}$ and $y \ge 9/2$

4. If $log\ x + log\ y = 2x$, express y as an explicit function of x:
   (i) $y = \frac{bx+d}{ax+c}$ (ii) $y = \frac{c^{2x}}{x}$
   (iii) $y = \frac{e^{2x}+2}{2x}$ (iv) $y = log_e (x \pm \sqrt{x^2-1})$

5. If $f(x) = 10x^2 - 13x + 13$, solve the equation $f(x) = 16$
   (i) 2, 5 (ii) $^2/_5, ^1/_3$ (iii) 3, 7 (iv) $^3/_2, ^{-1}/_5$

6. If $y = f(x) = \frac{3x-5}{2x-m}$ and $f(y) = x$, find the value of m.
(i) $m = 6$ (ii) $m = 3$ (iii) $m = 7$ (iv) $m = 8$

Lesson-2: Limit

After studying this lesson, you should be able to:

- Discuss the nature of fundamental theorems on limit;
- Apply the different methods of evaluating the limit.

Introduction

Limit determines whether the value of a function exists in the neighborhood of a point at which the function is undefined

The concept of limit is an operation, which determines whether the value of a function exists in the neighborhood of a point at which the function is undefined. It is completely new concept in mathematics and is considered to be the basis of calculus. Now-a-days, this concept has wide application in the theoretical discussion in different branches of science including mathematics and in the solution of different problems in economics. In this lesson we shall make a brief discussion on the limit of a function and the application of fundamental theorems in evaluating limit of a function.

Limit of a Function

Generally, we are concerned with what happens to the value of the dependent variable *f(x)* as the value of the independent variable *x* approaches some constant *a*. For example, the function *f* defined by $f(x) = x+2$ and notice what happens to the value of *f(x)* as the value of *x* moves closer and closer to 2.

Let us set up a table of *x* and corresponding *f(x)* values, as

x : 1.9 → 1.99 → 1.999 → 1.9999 2.0001 → 2.001 → 2.01 → 2.1

$f(x)$: 3.9 → 3.99 → 3.999 → 3.9999 4.0001 → 4.001 → 4.01 → 4.1

Fundamental Theorems of Evaluating Limit of a Function

The following theorems are most useful in the evaluation of limits.

For any real number *a*, assuming that $\lim_{x\to a} f(x)$ and lim $\lim_{x\to a} g(x)$ exist:

1. For any real constant *k*, $\lim_{x\to a} k = k$

2. For any real number n, $\lim_{x\to a} x^n = a^n$

3. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ in the root is defined

4. $\lim_{x\to a} k.f(x) = k.\lim_{x\to a} f(x)$

5. $\lim_{x\to a} [f(x) \pm g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

6. $\lim_{x\to a} f(x).g(x) = \lim_{x\to a} f(x).\lim_{x\to a} g(x)$

7. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \lim_{x\to a} f(x) / \lim_{x\to a} g(x)$; if $\lim_{x\to a} g(x) \neq 0$

8. If n is any positive integer, then

(a) $\lim_{x \to \infty^+} \frac{1}{x^n} = 0$ (b) $\lim_{x \to \infty^-} \frac{1}{x^n} = 0$

9. If n is any positive integer, then

$\lim_{x \to 0^+} \frac{1}{x^n} = +\infty$ and $\lim_{x \to 0^+} \frac{1}{x^n} = +\infty$ (if n is even) or $-\infty$ (if n is odd)

10. $\lim_{x \to a} \log f(x) = \log \lim_{x \to a} f(x)$

11. $\lim_{x \to a} [f(x)]^n = [\lim_{x \to a} f(x)]^n$

The following examples illustrate the uses of these theorems.

Example-1:

Compute $\lim_{x \to 0} \frac{\sqrt{3+x} - \sqrt{3-x}}{x}$

Solution:

$$\lim_{x \to 0} \frac{\sqrt{3+x} - \sqrt{3-x}}{x} \times \frac{\sqrt{3+x} + \sqrt{3-x}}{\sqrt{3+x} + \sqrt{3-x}}$$

$$= \lim_{x \to 0} \frac{(3+x)-(3-x)}{x.[\sqrt{3+x} + \sqrt{3-x}]} = \lim_{x \to 0} \frac{2}{\sqrt{3+x} + \sqrt{3-x}} = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}}$$

Example-2:

Evaluate $\lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$; where, $g(x) = 7x + 9$

Solution:

$$\lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$= \lim_{h \to 0} \frac{[7(x+h)+9] - [7x+9]}{h}$$

$$= \lim_{h \to 0} \frac{7x+7h+9-7x-9}{h} = \lim_{h \to 0} \frac{7h}{h}$$

$= \lim_{h \to 0} 7 = 7.$ The constant function 7 is continuous.

Example-3:

Evaluate $\lim_{x \to 0} \frac{e^x - e^{-x}}{x}$

Solution:

$$\lim_{x\to 0}\frac{e^x - e^{-x}}{x}$$

$$= \lim_{x\to 0}\frac{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots\dots\dots\right) - \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \dots\dots\right)}{x}$$

$$= \lim_{x\to 0}\frac{2\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots\dots\right)}{x}$$

$$= \lim_{x\to 0} 2\left(1 + \frac{x^2}{3!} + \frac{x^4}{5!} + \frac{x^6}{7!} + \dots\dots\right) = \lim_{x\to 0}(2\times 1) = 2$$

Example-4:

Prove that $\lim_{x\to 0}\frac{1-\cos x}{x} = 0$

Solution:

$$\lim_{x\to 0}\frac{1-\cos x}{x} = 0$$

$$= \lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{x} = \lim_{x\to 0}\frac{Sin^2\frac{x}{2}}{\frac{x}{2}}$$

$$= \lim_{x\to 0}\frac{\sin^2\frac{x}{2}}{\frac{x}{2}} \times \lim_{x\to 0}\sin\frac{x}{2} = (1\times 0) = 0$$

Example-5:

Find (a) $\lim_{x\to 2}(3x^2 - x + 6)$ (b) $\lim_{x\to 3}(2x^2+1)(3x-4)$

Solution:

(a) $\lim_{x\to 2}(3x^2 - x + 6)$

$= \lim_{x\to 2} 3x^2 - \lim_{x\to 2} x + \lim_{x\to 2} 6$

$= [3(2)^2 - 2 + 6] = (12 - 2 + 6) = 16$

(b) $\lim_{x\to 3}(2x^2+1)(3x-4)$

$= [\lim_{x\to 3} 2x^2 + \lim_{x\to 3} 1]\,[\lim_{x\to 3} 3x - \lim_{x\to 3} 4]$

$= [2.(3)^2 + 1].\,[3.(3) - 4]$

$= [(18+1).(5)] = (19 \times 5) = 95$

Example-6:

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$$

Solution:

Here Substituting $x = 2$, we get $\frac{0}{0}$ which does not exist.

Hence by rationalizing, $\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$

$= \lim_{x \to 2} \frac{(x+2)(x-2)}{(x-2)}$ [as x ≠ 2; ∴ x – 2 ≠ 0]

$= \lim_{x \to 2} (x+2) = (2 + 2) = 4$

$\therefore \lim_{x \to 2} \frac{x^2 - 4}{x - 2} = 4.$

Some Important Limits

The following formulae are also used for evaluating the limit of a function.

(1) $\lim_{x \to 0} \frac{x^n - a^n}{x - a} = n.a^{n-1}$

(2) $\lim_{n \to 0} (1+n)^{\frac{1}{n}} = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e.$

(3) $\lim_{n \to 0} \frac{(1+x)^n - 1}{x} = n$

(4) $\lim_{n \to 0} \frac{\sin x}{x} = 1$

Example-7:

Find (a) $\lim_{x \to 1} \left[2x^2(x + \sqrt{x}) + 3x^{\frac{1}{3}} - \frac{14}{x} \right]$ (b) $\lim_{x \to \infty} \frac{8x^2 + 16x + 3}{2x^3 - x + 3}$

Solution:

(a) $\lim_{x \to 1} \left[2x^2(x + \sqrt{x}) + 3x^{\frac{1}{3}} - \frac{14}{x} \right]$

$= (2 \lim_{x \to 1} x^2).(\lim_{x \to 1} x + \lim_{x \to 1} \sqrt{x}) + 3 \lim_{x \to 1} x^{1/3} - 14 \lim_{x \to 1} \frac{1}{x}$

$= 2\,(1^2)\,(1 + \sqrt{1}) + 3\,(1^{1/3}) - 14\left(\frac{1}{1}\right) = -7.$

(b) $\lim_{x\to\infty}\frac{8x^2+16x+3}{2x^3-x+3}$

$= \lim_{x\to\infty}\frac{\frac{8x^2+16x+3}{x^3}}{\frac{2x^3-x+3}{x^3}}$

$= \lim_{x\to\infty}\frac{\frac{8}{x}+\frac{16}{x^2}+\frac{3}{x^3}}{2-\frac{1}{x^2}+\frac{3}{x^3}}$

$= \frac{0}{2}\left[where,\ \lim_{x\to\infty}\frac{1}{x}=0\right]$

$= 0.$

Example-8:

Show that, $\lim_{x\to 2}\left(x^2-3x+5\right)=3$

Solution:

We have $\lim_{x\to 2}\left(x^2-3x+5\right)=3$

$= \lim_{x\to 2} x^2 - \lim_{x\to 2} 3x + \lim_{x\to 2} 5$

$= \lim_{x\to 2} x.\ \lim_{x\to 2} x - 3\ \lim_{x\to 2} x + 5$

= (2 × 2 – 3 × 2 + 5) = (4 – 6 + 5) = 3 (Proved)

Questions for Review

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. Give the definition of limits of a function.
2. Mention the fundamental theorems of evaluating a function.
3. Find $\lim_{x \to 2} (3x^2 + 2)$
4. If $f(x) = \frac{1}{x}$, find $\lim_{x \to 0} f(x)$
5. Evaluate $\lim_{x \to 1} \frac{x^2 - 3x + 2}{x - 1}$
6. Evaluate $\lim_{x \to \infty} \frac{2x + 3}{x + 1}$
7. Evaluate $\lim_{x \to 0} \left\{ \frac{\sqrt{1+x} - \sqrt{1-x}}{x} \right\}$
8. Prove that $\lim_{x \to 4} \log \left(2x^{\frac{3}{2}} - 3x^{\frac{1}{2}} - 1 \right) = 2 \log 3.$
9. Evaluate $\lim_{x \to -1} \frac{2x^2 - x - 3}{x^2 - 2x - 3}$
10. Find the value of $\lim_{x \to 0} \frac{\sqrt{1+2x} - \sqrt{1-3x}}{x}$

Multiple Choice Questions (Tick the most appropriate answer)

1. Find the value of $\lim_{x \to a} \frac{\sqrt{x} - \sqrt{a}}{x - a}$

   i) $\frac{1}{\sqrt{a}}$ ii) $\frac{1}{2\sqrt{a}}$ iii) $2\sqrt{a}$ iv) $\sqrt{a}$

2. Evaluate $\lim_{x \to 2} \frac{x^2 - 5x + 6}{x^3 - 5x + 2}$

   i) $\frac{1}{5}$ ii) $\frac{1}{-3}$ iii) $\frac{1}{-7}$ iv) -7

3. Evaluate $\lim_{x \to 0} \frac{\sqrt{1-x^3} - \sqrt{1-x}}{\sqrt{1+x^2} - \sqrt{1+x}}$

   i) -1 *ii)* -6 *iii)* 1 *iv)* $\frac{-1}{2}$

4. Evaluate $\lim_{x \to -1} \dfrac{e^{\log x} - 1}{e^{x-1} - 1}$

i) $\dfrac{5}{3}$ ii) $\dfrac{2}{7}$ iii) 1 iv) 2

5. If $g(x) = -\sqrt{25 - x^2}$, find the value of $\lim_{x \to 1} \dfrac{g(x) - g(1)}{x - 1}$

i) $\dfrac{1}{2}$ ii) $\dfrac{1}{2\sqrt{6}}$ iii) $\dfrac{1}{\sqrt{6}}$ iv) $\dfrac{1}{2}$

6. If $\lim_{x \to 2} \dfrac{ax^2 - b}{x - 2} = 4$, find the values of *a* and *b*.

i) $a = 2, b = 3$ ii) $a=1, b=3$ iii) $a=1, b=4$ iv) $a=6, b=4$.

Lesson-3: Continuity

After studying this lesson, you should be able to:

- Discuss the nature of continuity of a function;
- Apply the conditions for continuity of a function.

Nature of Continuity

A function $f(x)$ is said to be continuous in an open or closed interval if it is continuous at all points in the interval. *For example*, the function $f(x) = x^2$ is continuous in the closed interval $-4 \leq x \leq 3$ when it is continuous at every point in the interval.

A function f(x) is said to be continuous in an open or closed interval if it is continuous at all

If the function $f(x)$ is not continuous at $x = a$, we say that the function $f(x)$ is discontinuous at $x = a$ and the point $x = a$ is called a point of discontinuity of the function. The function $f(x)$ is said to be discontinuous at $x = a$ if,

(i) $f(a)$ is undefined i.e. $f(x)$ does not possess a definite finite value at $x = a$

or, (ii) $\lim\limits_{x \to a} f(x)$ does not exist

or, (iii) $\lim\limits_{x \to a} f(x)$ exists but $\lim\limits_{x \to a} f(x) \neq f(a)$

Continuity of a Function

The important concept of continuity of a function is developed from the theory of limit. A function '*f*' is continuous at $x = a$ if and only if all of the following conditions apply to f at a.

1) $f(a)$ is defined, i.e., the domain of f includes $x = a$;

2) $\lim\limits_{x \to a} f(x)$ exists;

3) $\lim\limits_{x \to a} f(x) = f(a)$, whether x approaches to a from the left or from the right.

Continuity at an Interval

If a and b are real numbers and $a < b$, then the set $\{x \mid a < x < b\}$ is called an open interval and is denoted by $[a, b]$. The set $\{x \mid a \leq x \leq b\}$ is called a closed interval and denoted by (a, b). The half-open interval $\{x \mid a \leq x < b\}$ is symbolized (a, b] whereas the half-closed interval $\{x \mid a < x \leq b\}$ is symbolized [a, b). In each case a and b are the endpoints of the interval, and any x value such that $a < x < b$ is interior point.

A function f is continuous on an open interval if it is continuous at each number in that interval.

A function f is continuous on an open interval if it is continuous at each number in that

A function f is continuous on a closed interval (a, b) provided the following conditions are satisfied:

1. f is continuous over the open interval (a, b)
2. $f(x) \to f(a)$ as $x \to a$ from within (a, b)
3. $f(x) \to f(b)$ as $x \to b$ from within (a, b)

The following examples illustrate the requirements/conditions for continuity of a function.

Example-1:

Show that $f(x) = \dfrac{x^2 - 4}{x - 2}$ is not continuous $x = 2$ but continuous at $x = 3$.

Solution:

The conditions to be satisfied by a function before we can say that it is continuous at a particular point say $x = a$ are: $f(a)$, $\lim_{x \to a-} f(x)$ and $\lim_{x \to a+} f(x)$ should have definite and finite values and these are all equal.

Let us examine whether these conditions are satisfied by $f(x) = \dfrac{x^2 - 4}{x - 2}$ for $x = 2$.

Here $x = 2$, therefore we have

(i) $f(2) = \dfrac{2^2 - 4}{2 - 2} = \dfrac{0}{0}$, which is undefined.

Again by the method of finding the left hand and right hand side limits, we have

(ii) $\lim_{x \to 2^-} \dfrac{x^2 - 4}{x - 2} = \lim_{x \to 2^-} \dfrac{(x + 2)(x - 2)}{(x - 2)}$

$= \lim_{x \to 2^-} (x + 2)$

$= \lim_{h \to 0} (2 - h + 2) = 4$

$\therefore$ L.H.S. limit = 4.

Again, $\lim_{x \to 2^+} \dfrac{x^2 - 4}{x - 2} = \lim_{x \to 2^+} \dfrac{(x + 2)(x - 2)}{(x - 2)}$

$= \lim_{x \to 2^+} (x + 2)$

$= \lim_{h \to 0} (2 + h + 2) = 4$

$\therefore$ R.H.S. limit = 4.

Here, $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) \neq f(2)$

$\therefore f(x) = \dfrac{x^2 - 4}{x - 2}$ is not continuous at $x = 2$.

Now, for $x = 3$,

i) $f(3) = \dfrac{3^2 - 4}{3 - 2} = 5$ and

ii) $\lim_{x\to 3^-} \frac{x^2-4}{x-2} = \lim_{x\to 3^-} \frac{(x+2)(x-2)}{(x-2)}$

$= \lim_{h\to 0}(3 - h + 2) = 5$

$\therefore$ L.H.S. limit = 5.

Again, $\lim_{x\to 3^+} \frac{x^2-4}{x-2} = \lim_{x\to 3^+} \frac{(x+2)(x-2)}{(x-2)}$

$= \lim_{h\to 0}(3 + h + 2) = 5$

$\therefore$ R.H.S. limit = 5.

Here, $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) \neq f(3)$

$\therefore f(x) = \frac{x^2-4}{x-2}$ is continuous at $x = 3$.

Example-2:

Show that $f(x) = 3x^2 + 2x - 1$ is continuous at $x = 2$. Also prove that $f(x)$ is continuous for all values of x.

Solution:

The condition is to be satisfied by a function if we can say that it is continuous at a particular point say $x = a$, where $\lim_{x\to a^-} f(x) = f(a)$ $= \lim_{x\to a^+} f(x)$

Let us examine whether these conditions are satisfied by $f(x) = 3x^2 + 2x - 1$ for $x = 2$.

Here $a = 2$, therefore, we have (i) $f(2) = (3.\ 2^2 + 2.2 - 1) = 15$.

Again by the method of finding the left and right hand side limit, we have

(ii) $\lim_{x\to 2^-} (3x^2 + 2x - 1) = \lim_{h\to 0} \{(3(2 - h)^2 + 2(2 - h) - 1\} = 15$

$\therefore$ L.H.S. limit = 15.

Again (iii) $\lim_{x\to 2^+} (3x^2 + 2x - 1) = \lim_{h\to 0} \{(3(2 + h)^2 + 2(2 + h) - 1\} = 15$

$\therefore$ R.H.S. limit = 15.

We find the values of the function at $x = 2$, the LHS limit and RHS limit and all of these exist and finite and equal. Thus the $f(x) = 3x^2 + 2x - 1$ is continuous at $x = 2$.

We shall show further that $f(x) = 3x^2 + 2x - 1$ is continuous for all values of x.

Let $x = k$ be any value of x arbitrarily selected and find out whether given function is continuous at $x = k$

Here $a = k$, therefore $f(k) = 3k^2 + 2k - 1$ (finite number) .... (1)

Also, $\lim_{x\to k^-} (3x^2 + 2x - 1) = \lim_{h\to 0} \{(3(k-h)^2 + 2(k-h) - 1\}$

$$= \lim_{h\to 0} (3k^2 - 6kh + 3h^2 - 2k + 2h - 1)$$

$= (3k^2 + 2k - 1)$ ....(2)

Similarly we find that, $\lim_{x\to k^+} (3x^2 + 2x - 1) = 3k^2 + 2k - 1$ .... (3)

From (1), (2) and (3) we deduce that the given function is continuous at $x = k$.

Since k is any arbitrary value of x, therefore, $f(x)$ is continuous for all values of x.

Example-3:

Find the points of discontinuity of the function $\dfrac{x^2 - 3x - 4}{x^3 - 2x^2 - 5x + 6}$

Solution:

Let $f(x) = \dfrac{x^2 - 3x - 4}{x^3 - 2x^2 - 5x + 6}$

We know that if a function is undefined at $x = a$, then $x = a$ is a point of discontinuity of the function. Therefore, the points of discontinuity of $f(x)$ are the values of x at which $f(x)$ becomes undefined. The values of x for which $f(x)$ is undefined are the roots of the equation.

$x^3 - 2x^2 - 5x + 6 = 0$

or, $x^2(x - 1) - x(x - 1) - 6(x - 1) = 0$

or, $(x - 1)(x^2 - x - 6) = 0$

or, $(x - 1)(x^2 - 3x + 2x - 6) = 0$

or, $(x - 1)[x(x - 3) + 2(x - 3)] = 0$

or, $(x - 1)(x + 2)(x - 3) = 0$

$\therefore$ $x = 1$ or $x = -2$ or $x = 3$

Hence, the points of discontinuity of $f(x)$ are : $x = 1$, $x = 3$ and $x = -2$.

Example-4:

The function $f(x) = \dfrac{2x^2 - 8}{x - 2}$ is undefined at $x = 2$. What value must be assigned to $f(2)$, if $f(x)$ is to be continuous at $x = 2$.

Solution:

We have, $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \dfrac{2(x^2 - 4)}{(x - 2)}$

$$= \lim_{x\to 2^+} \frac{2(x + 2)(x - 2)}{(x - 2)}$$

$$= \lim_{h \to 0} [2(2 + h + 2)] = 2(2+2) = 8.$$

Similarly, $\lim_{x \to 2^-} f(x) = 8$

$\therefore \lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x) = 8$

Therefore, $\lim_{x \to 2} f(x) = 8$

Now, the function $f(x)$ will be continuous at $x = 2$ if $\lim_{x \to 2} f(x) = f(2)$; i.e., if $f(2) = 8$.

Hence the required assigned value of $f(2)$ is 8.

Questions for Review

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. Define the continuity of $f(x)$ at $x = a$. When is the function said to be continuous in the closed interval $a \leq x \leq b$?
2. Define the discontinuity of $f(x)$ at $x = a$.
3. Indicate the points of discontinuity of the function: $\frac{2x^2+6x-5}{12x^2+x-20}$.
4. The function $f(x) = \frac{x^3-8}{x^2-4}$ is undefined at $x = 2$. Redefine the function so as to make it continuous at $x = 2$.
5. It $f(x) = \frac{x^2-9}{x-3}$ when $x \neq 3$; state the value of $f(3)$ so that $f(x)$ is continuous at $x = 3$.

Differentiation and its Uses in Business Problems

8

The objectives of this unit is to equip the learners with differentiation and to realize its importance in the field of business. The unit surveys derivative of a function, derivative of a multivariate functions, optimization of lagrangian multipliers and Cobb-Douglas production function etc. Ample examples have been given in the lesson to demonstrate the applications of differentiation in practical business contexts. The recognition of differentiation in decision making is extremely important in the filed of business.

Blank Page

Lesson-1: Differentiation

After studying this lesson, you should be able to:

- Explain the nature of differentiation;
- State the nature of the derivative of a function;
- State some standard formula for differentiation;
- Apply the formula of differentiation to solve business problems.

Introduction

Calculus is the most important ramification of mathematics. The present and potential managers of the contemporary world make extensive uses of this mathematical technique for making pregnant decisions. Calculus is inevitably indispensable to measure the degree of changes relating to different managerial issues. Calculus makes it possible for the enthusiastic and ambitious executives to determine the relationship of different variables on sound footings. Calculus in concerned with dynamic situations, such as how fast production levels are increasing, or how rapidly interest is accruing.

The term calculus is primarily related to arithmetic or probability concept. Mathematics resolved calculus into two parts - differential calculus and integral calculus. Calculus mainly deals with the rate of changes in a dependent variable with respect to the corresponding change in independent variables. Differential calculus is concerned with the average rate of changes, whereas Integral calculus, by its very nature, considers the total rate of changes in variables.

Differential calculus is concerned with the average rate of changes.

Differentiation

Differentiation is one of the most important operations in calculus. Its theory solely depends on the concepts of limit and continuity of functions. This operation assumes a small change in the value of dependent variable for small change in the value of independent variable. In fact, the techniques of differentiation of a function deal with the rate at which the dependent variable changes with respect to the independent variable. This rate of change is measured by a quantity known as derivative or differential co-efficient of the function. Differentiation is the process of finding out the derivatives of a continuous function i.e., it is the process of finding the differential co-efficient of a function.

The techniques of differentiation of a function deal with the rate at which the dependent variable changes with respect to the independent

Derivative of a Function

The derivative of a function is its instantaneous rate of change. Derivative is the small changes in the dependent variable with respect to a very small change in independent variable.

Let $y = f(x)$, derivative i.e. $\frac{dy}{dx}$ means rate of change in variable y with respect to change in variable x.

The derivative has many applications, and is extremely useful in optimization- that is, in making quantities as large (for example profit) or as small (for example, average cost) as possible.

Some Standard Formula for Differentiation

Following are the some standard formula of derivatives by means of which we can easily find the derivatives of algebraic, logarithmic and exponential functions. These are :

1. $\frac{dc}{dx} = 0$, where C is a constant.

2. $\frac{dx^n}{dx} = \frac{d}{dx}[x^n] = n.\, x^{n-1}$

3. $\frac{d}{dx} a.f(x) = a\frac{d}{dx}[f(x)]$

4. $\frac{d}{dx}\left(x^{-\frac{1}{n}}\right) = -\frac{1}{n}.x^{\frac{-(n+1)}{n}}$

5. $\frac{de^x}{dx} = \frac{d}{dx}(e^x) = e^x$

6. $\frac{d}{dx}\left[e^{g(x)}\right] = e^{g(x)}.\frac{d}{dx}[g(x)]$

7. If $y = f(u)$ and $U = g(x)$ then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

8. $\frac{d}{dx}(a^x) = a^x.\log_e a$

9. $\frac{d[f(x) \pm g(x)]}{dx} = \frac{d[f(x)]}{dx} \pm \frac{d[g(x)]}{dx}$

10. $\frac{d}{dx}(\log_e x) = \frac{d}{dx}(\ln \quad x) = \frac{1}{x}$

11. If $Y = [f(x)]^n$ then, $\frac{dy}{dx} = n\,[f(x)]^{n-1}\,.\frac{d[f(x)]}{dx}$

12. $\frac{d}{dx}\log_a x = \frac{1}{x}\ \log_a e$

13. $\frac{d[f(x).g(x)]}{dx} = f(x)\frac{d[g(x)]}{dx} + g(x)\frac{d[f(x)]}{dx}$

14. $\dfrac{d\left[\dfrac{f(x)}{g(x)}\right]}{dx} = \dfrac{g(x)\dfrac{d[f(x)]}{dx} - f(x)\dfrac{d[g(x)]}{dx}}{[g(x)]^2}$

15. $\dfrac{d}{dx}a^{g(x)} = a^{g(x)}.\dfrac{d}{dx}[g(x)].\log_a e$

16. If $U = f(x, y)$, $\dfrac{du}{dx} = \left[\dfrac{f(x + dx.y) - f(x, y)}{dx}\right]$ and

$\dfrac{du}{dy} = \left[\dfrac{f(x, y + dy) - f(x, y)}{dy}\right]$

17. If y = e^{ax}, then its first derivative is equal to $\dfrac{de^{ax}}{dx}$ = e^{ax}

Second derivative is equal to $\dfrac{d^2e^{as}}{dx^2}$ = a^2e^{ax}

Third derivative is equal to $\dfrac{d^3e^{ax}}{dx3}$ = a^3e^{ax} and the nth derivative is denoted by

$\dfrac{d^ne^{ax}}{dx^n}$ = a^ne^{ax}

Derivative of Trigonometrically Functions

18. $\dfrac{d}{dx}$(Sin x) = Cos x. $\dfrac{d}{dx}$ (Cos x) = – Sinx

19. $\dfrac{d}{dx}$(tan x) = Sec^2x. $\dfrac{d}{dx}$ (Cot x) = – $Cosec^2$x

20. $\dfrac{d}{dx}$ (Sec *x*) = Sec *x*. tan *x*; $\dfrac{d}{dx}$(Cosec *x*) = – Cosec *x*. Cot *x*

21. $\dfrac{d}{dx}\left(Sin^{-1}x\right) = \dfrac{1}{\sqrt{1-x^2}}$; $\dfrac{d}{dx}\left(Cos^{-1}x\right) = -\dfrac{1}{\sqrt{1-x^2}}$

22. $\dfrac{d}{dx}\left(\tan^{-1}x\right) = \dfrac{1}{1+x^2}$; $\dfrac{d}{dx}\left(Cot^{-1}x\right) = -\dfrac{1}{1+x^2}$

23. $\dfrac{d}{dx}\left(Sec^{-1}x\right) = \dfrac{1}{x.\sqrt{x^2-1}}$; $\dfrac{d}{dx}\left(Co\sec^{-1}x\right) = -\dfrac{1}{x\sqrt{x^2-1}}$

$Sin^2x + Cos^2x = 1$; tan x = $\dfrac{\text{Sin x}}{\text{Cos x}}$

$Sec^2 x - \tan^2 x = 1$; $\text{Cot x} = \frac{\text{Cos x}}{\text{Sin x}}$

When x and y are separately expressed as the functions of a third variable in the equation of a curve is known as parameter. In such cases we can find $\frac{dy}{dx}$ without first eliminating the parameter as follows:

Thus, if x =Q (t), y = ψ (t)

Then, $\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Let us illustrate these different derivatives by the following examples.

Example – 1:

If y = f(x) = a; find $\frac{dx}{dy}$

Solution:

$\frac{dy}{dx} = \frac{d(a)}{dx}$ =0, since a is a constant, i.e.,'a' has got no relationship with variable x.

Example – 2:

Differentiate the following functions, with respect to x,

(i) $y = \sqrt{x}$, (ii) $y = 8x^{-5}$ (iii) $y= 3x^3 - 6x^2+2x - 8$

Solution:

We know that $\sqrt{x} = x^{\frac{1}{2}}$

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^{\frac{1}{2}}\right) = \frac{1}{2}.x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}} \text{ . Hence } \frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$

(ii) $\frac{dy}{dx} = \frac{d}{dx}(8x^{-5})$

$$= 8\frac{d}{dx}(x^{-5}) = 8(-5)\,x^{-6} = -40x^{-6} = \frac{-40}{x^6}\ .$$

Therefore, $\frac{dy}{dx} = \frac{-40}{x^6}$

(iii) $\frac{dy}{dx} = \frac{d}{dx}(3x^3 - 6x^2 + 2x - 8)$

$= \frac{d}{dx}(3x^3) - \frac{d}{dx}(6x^2) + \frac{d}{dx}(2x) - \frac{d}{dx}(8)$

$= 3.3x^{3-1} - 2.6.x^{2-1} + 2 - 0 = 9x^2 - 12x + 2$

Thus, $\frac{dy}{dx} = 9x^2 - 12x + 2.$

Example –3:

Differentiate $e^x(\log x) . (2x^2+3)$ with respect to x.

Solution:

Let $y = e^x.(\log x) . (2x^2 + 3)$

$\frac{dy}{dx} = e^x (\log x)\frac{d}{dx}(2x^2+3) + e^x(2x^2+3).\frac{d}{dx}(\log x) + (\log x).(2x^2+3).\frac{d}{dx}(e^x)$

$= e^x(\log x)(4x) + e^x(2x^2+3)\frac{1}{x} + \log x(2x^2+3)e^x$

$= e^x[4x.\log x + \frac{2x^2+3}{x} + (2x^2+3)\log x]$

So, $\frac{dy}{dx} = e^x[4x.\log x + \frac{2x^2+3}{x} + (2x^2+3)\log x]$

Example–4:

If $y = \frac{2+3\log x}{x^2+5}$, find $\frac{dy}{dx}$

Solution:

$y = \frac{2+3\log x}{x^2+5}$

$$\frac{dy}{dx} = \frac{(x^2+5)\frac{d}{dx}(2+3\log x) - (2+3\log x)\frac{d}{dx}(x^2+5)}{(x^2+5)^2}$$

$= \frac{(x^2+5)(\frac{3}{x}) - (2+3\log x)(2x)}{(x^2+5)^2} = \frac{\frac{15}{x} - x - 6x\log x}{(x^2+5)^2}$. Thus, $\frac{dy}{dx} = \frac{\frac{15}{x} - x - 6x\log x}{(x^2+5)^2}$

Example–5:

Find the differential co-efficient of e^{x^2+5x+7} with respect to x.

Solution:

Let y= e^{x^2+5x+y}

$$\frac{dy}{dx} = e^{x^2+5x+y} \cdot \frac{d}{dx}(x^2 + 5x + y) = e^{x^2+5x+y} \cdot (2x+5)$$

$$\frac{dy}{dx} = (2x+5). e^{x^2} + 5x + y$$

Example–6:

Find $\frac{dy}{dx}$, if y = $\log\sqrt{4x+3}$

Solution:

Let y = $\log\sqrt{4x+3}$

$$= \log(4x+3)^{\frac{1}{2}} = \frac{1}{2}\log(4x+3)$$

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{1}{2}\log(4x+3)\right] = \frac{1}{2} \cdot \frac{1}{4x+3} \cdot \frac{d}{dx}(4x+3) = \frac{1}{2} \cdot \frac{1}{4x+3} \cdot 4 = \frac{2}{4x+3}$$

$$\therefore \frac{dy}{dx} = \frac{2}{4x+3}$$

Example–7:

Find the first, second and third derivatives when y = $x.e^{x^2}$

Solution:

y = $x.e^{x^2}$

First derivative,

$$\frac{dy}{dx} = x\frac{d}{dx} e^{x^2} + e^{x^2}.\frac{d}{dx}(x) = x.e^{x^2}.2x + e^{x^2}.1 = e^{x^2}(2x^2+1)$$

Second derivative, $\frac{d^2y}{dx^2} = e^{x^2}.2x(2x^2+1) + e^{x^2}.4x$

$$= e^{x^2}.4x^3 + e^{x^2}.2x + e^{x^2}.4x = e^{x^2}.(4x^3+2x+4x)$$

$$= e^{x^2}.(4x^3+6x)$$

Third derivative,

$$\frac{d^3y}{dx^3} = e^{x^2}\, 2x\, (4x^3 + 6x) + e^{x^2}\, (12x^2 + 6)$$

$$= e^{x^2}.\, 8x^4 + e^{x^2}\, 12x^2 + e^{x^2}\, (12x^2 + 6) = e^{x^2}\, (8x^4 + 12x^2 + 12x^2 + 6)$$

$$= e^{x^2}\, (8x^4 + 24x^2 + 6)$$

Example–8:

If $y = x^{\log x}$, find $\frac{dy}{dx}$

Solution:

Given, $y = x^{\log x}$

Taking logarithm of both sides, we have

$\log y = \log (x^{\log x}) = \log x.\, \log x = (\log x)^2$

Differentiating with respect to x, we get

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{d}{dx} (\log x)^2 = 2\, (\log x)^{2-1} \frac{d}{dx} (\log x) = 2 \log x.\, \frac{1}{x}$$

Hence, $\frac{dy}{dx} = y\, (2 \log x.\, \frac{1}{x}) = \frac{2x^{\log x}.\, \log x}{x}$

Example–9:

Find the derivative of log (ax + b) with respect to x.

Solution:

Let $y = \log (ax + b)$

So, $\frac{dy}{dx} = \frac{d}{dx}\, [\log (ax + b)] = \frac{1}{(ax+b)} \frac{d}{dx} (ax + b) = \frac{a}{ax+b}$.

Therefore, $\frac{dy}{dx} = \frac{a}{ax + b}$

Example–10:

Differentiate $y = \log_a x$ with respect to x.

Solution:

We know that $\log_a x = \frac{\log_e x}{\log_e a}$

So, $\frac{dy}{dx}=\frac{d}{dx}\left(\log_a x\right)=\frac{d}{dx}\left(\frac{\log_e x}{\log_e a}\right)$

$=\frac{1}{\log_e a}.\frac{d}{dx}\log_e x=\frac{1}{\log_e a}.\frac{1}{x}=\frac{1}{x.\log_e a}$

So, $\frac{dy}{dx}=\frac{1}{x.\log_e a}$

Example–11:

If $y = x^{x^x}$, find $\frac{dy}{dx}$

Solution:

$y = x^{x^x}$

Taking logarithm of both sides, we have

$\log y = \log x^{x^x} = x^x \log x.$

Differentiating with respect to x, we get,

$\frac{1}{y} \cdot \frac{dy}{dx} = x^x \frac{d}{dx}(\log x) + \log x \frac{d}{dx}(x^x)$

or, $\frac{1}{y} \cdot \frac{dy}{dx} = x^x \cdot \frac{1}{x} + \log dx^x (1+\log x)$

$\therefore \frac{dy}{dx} = y\, [x^x \cdot \frac{1}{x} + \log x^x (1+\log x)]$

$= x^{x^x} [x^{x-1} + \log x^x (1+\log x)]$

Let $y = x^x$

Then $\log y = x \log x$

$\frac{d}{dx}(\log y) = 1(\log x) + x.\frac{1}{x}$

$= 1 + \log x$

$\frac{1}{y}\frac{dy}{dx} = 1 + \log x$

So, $\frac{dy}{dx} = y\,(1+\log x)$

So, $\frac{dy}{dx} = x^x (1+ \log x)$

Example–12:

Differentiate $y = \log [\text{Sin } (3x^2+5)]$ with respect to x.

Solution:

$y = \log [\text{Sin } (3x^2+5)]$

$\frac{dy}{dx} = \frac{d}{dx} [\log \{\text{Sin } (3x^2+5)\}]$

$= \frac{1}{\text{Sin}(3x^2+5)} . \frac{d}{dx} [\text{Sin } (3x^2+5)]$

$= \frac{1}{\text{Sin}(3x^2+5)} . \text{Cos } (3x^2+5) . \frac{d}{dx}(3x^2+5)$

$$= \frac{1}{\text{Sin}(3x^2+5)} \cdot \text{Cos}(3x^2+5) \cdot 6x = 6x.\frac{Cos\left(3x^2+5\right)}{Sin\left(3x^2+5\right)} = 6x.\text{Cot}\ (3x^2+5)$$

So, $\frac{dy}{dx} = 6x.\text{Cot}\ (3x^2+5)$

Questions for Review:

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions :

1. Define differentiation. What are the fundamental theorems of differentiation?

2. Why is the study of differentiation important in managerial decision making?

3. Find the derivative of the following functions with respect to x.

   i) $5x^4+\frac{3}{x5} - 8x^2+7x$, (ii) $(2x^3-5x^{-2}+2)(4x^2-3\sqrt{x})$ (iii) $\frac{5x^2+9}{3x-2}$

   (iv) $(3x^2-2x+5)^{3/2}$ (v) $5e^x\log x$, (vi) $x^2+3xy+y^3=5$ (vii) e^{x^x}

4. If $y=x^3 \log n$, show that $\frac{d^4y}{dx^4}=\frac{6}{x}$

5. Differentiate the following w. r. to x

   (i) $\text{Sin } x^{\text{Cos } x}$ (ii) $\frac{\sin x}{\cos x}$ (iii) $e^{4x}+\log \sin x$

   iv) $(\sin^{-1}x)^{\log x}$

6. If $y = 8x^3-5x^{3/2}+3x^2-7x+5$: find $\frac{d^3y}{dx^3}$

Multiple Choice Questions (√ the most appropriate answer)

1. Find $\frac{dy}{dx}$ when $y = \log_2 x$

   i) $\log_2 e$ (ii) $\frac{1}{2}\log_2 e$ (iii) $2\log_2 e$ (iv) $\log e$

2. Find the derivatives of the function $\frac{1+x}{1-x}$

   i) $\frac{2}{1-x}$ (ii) $\frac{1}{(1-x)^2}$ (iii) $\frac{2}{(1-x)^2}$ (iv) $(1-x)^2$

3. If $f(x) = x^3 - 2px^2 - 4x + 5$ and $f(2) = 0$, find p.

   i) 1 (ii) 3 (iii) 4 (iv) 5

4. If $f(x)=2x^3-3x^2+4x-2$, find the value of $\frac{d}{dx}f(x=-2)$

   i) 30 (ii) 40 (iii) 25 (iv) 35

5. If $f(x)=\sqrt{2x} - \sqrt{\frac{2}{x}} + \frac{x+4}{4-x}$, find $\frac{d}{dx}f(x=2)$

   i) 2.75 (ii) 2 (iii) 2.5 (iv) 2.45

6. Find the derivative of x^x

   i) $x^n(1+\log n)$ (ii) $x \log n$ (iii) $x^2 \log n$ (iv) $x^2(1+\log n)$

7. If $y = 8x^3 - 5x^{3/2} + 3x^2 - 7x + 5$. find $\frac{d^2y}{dx^2}$

   i) $24x^2 - \frac{15}{2}x^{1/2} + 6x - 7$ (ii) $48x - \frac{15}{2}x^{1/2} + 6$

   (iii) $48 + \frac{15}{8\sqrt{x3}}$ (iv) $48x - \frac{15}{2\sqrt{x}} + 7$

Lesson-2: Differentiation of Multivariate Functions

Lesson Objectives:

After studying this lesson, you will be able to:

- State the nature of multivariate function;
- Explain the partial derivatives;
- Explain the higher- order derivatives of multivariate functions;
- Apply the techniques of multivariate function to solve the problems.

Introduction

The concept of the derivative extends directly to multivariate functions. During the discussion of differentiation, we defined the derivative of a function as the instantaneous rate of change of the function with respect to independent variable. In multivariate functions, there are more than one independent variable involved and thereby, the derivative of the function must be considered separately for each independent variable. For example, $z = f(x, y)$ is defined as a function of two independent variables if there exists one and only one value of z in the range of f for each ordered pair of real number (x, y) in the domain of f. By convention, z is the dependent variable; x and y are the independent variables.

To measure the effect of a change in a single independent variable (x or y) on the dependent variable (z) in a multivariate function, the partial derivative

To measure the effect of a change in a single independent variable (x or y) on the dependent variable (z) in a multivariate function, the partial derivative is needed. The partial derivative of z with respect of 'x' measures the instantaneous rate of change of z with respect to x while y is held constant. It is written $\frac{dz}{dx}, \frac{df}{dx}$, $f_x\ (x, y)$, f_x or Zx. The partial derivative of z with respect to y measures the rate of change of z with respect to y while x is held constant. It is written as: $\frac{dz}{dy}, \frac{df}{dy}$, $f_y\ (x, y)$, f_y or Zy.

Mathematically it can be expressed in the following way:

$$\frac{dz}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

$$\frac{dz}{dy} = \lim_{\Delta x \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

Partial differentiation with respect to one of the independent variables follows the same rules as differentiation while the other independent variables are treated as constant.

This is illustrated by the following examples.

Example–1:

Find the partial derivatives of M = f (x, y, z) = $x^2+5y^2+20xy+4z$.

Solution:

To determine the partial derivative of f with respect to *x*, we treat *y* and *z* as constants.

$$\frac{dm}{dx} = f_x{}' = 2x + 20y.$$

Similarly, in determining the partial derivative of f with respect to *y*, we treat *x* and *z* as a constants. $\frac{dm}{dy} = f_y{}' = 10y + 20x$

Finally, treating *x* and *y* as constants, we obtain the partial derivative of *f* with respect to *z* $\frac{dm}{dz} = f_z{}' = 4$

The same procedure is applied in the following examples.

Example – 2:

Determine the partial derivatives of

$Z = 5x^3 - 3x^2y^2 + 7y^5$

Solution:

$$\frac{dz}{dx} = \frac{d}{dx}(5x^3) - 3y^2\frac{d}{dx}(x^2) + \frac{d}{dx}(7y^5)$$

$$= 15x^2 - 6xy^2 + 0$$

$$\therefore \frac{dz}{dx} = 15x^2 - 6xy^2$$

Again $\frac{dz}{dy} = \frac{d}{dy}(5x^3) - 3x^2\frac{d}{dy}(y^2) + \frac{d}{dx}(7y^5)$

$$= 0 - 6x^2y + 35y^4$$

$$\therefore \frac{dz}{dy} = 35y^4 - 6x^2y.$$

Example–3:

Find the partial derivatives of $z = (5x + 3)(6x + 2y)$

Solution:

$$\frac{dz}{dx} = (5x + 3).\frac{d}{dx}(6x+2y) + (6x + 2y).\frac{d}{dx}(5x+3)$$

$$= (5x + 3).\ 6 + (6x + 2y).\ 5$$

$= 30x + 18 + 30x + 10y$

$\therefore \frac{dz}{dx} = 60x + 10y + 18$

Again $\frac{dz}{dy} = (5x + 3). \frac{d}{dy}(6x+2y) + (6x + 2y). \frac{d}{dx}(5x+3)$

$= (5x + 3). 2 + (6x + 2y). 0$

$\therefore \frac{dz}{dy} = 10x + 6 + 0 = 10x + 6$

Example–4:

Determine the partial derivatives of $Z = \frac{6x+7y}{5x+3y}$

Solution:

$$\frac{dz}{dx} = \frac{(5x+3y)\frac{d}{dx}(6x+7y) - (6x+7y)\frac{d}{dx}(5x+3y)}{(5x+3y)^2}$$

$$= \frac{(5x+3y). 6-(6x+7y).5}{(5x+3y)^2}$$

$$\frac{dz}{dx} = \frac{30x+18y-30x-35y}{(5x+3y)^2}$$

$$\therefore \frac{dz}{dx} = \frac{-17y}{(5x+3y)^2}$$

Again $\frac{dz}{dy} = \frac{(5x+3y). \frac{d}{dy}(6x+7y)-(6x+7y). \frac{d}{dy}(5x+3y)}{(5x+3y)^2}$

$$= \frac{(5x+3y).7 - (6x+7y).3}{(5x+3y)^2}$$

$$= \frac{35x+21y-18x-21y}{(5x+3y)^2}$$

$$\therefore \frac{dz}{dy} = \frac{17x}{(5x+3y)^2}$$

Higher-Order derivatives of Multivariate Functions

The rules for determining higher-order derivatives of functions of one independent variable apply to multivariate functions. Derivatives of multivariate functions are taken with respect to one independent variable at a time, the remaining independent variables being considered as

Derivatives of multi variate functions ar taken with respect t one independent variable at a time constants. The same procedure applies in determining higher-order derivatives of multivariate functions.

For instance, a function *f (x, y)* may have four second-order partial derivatives as follows:

Original function First partial derivatives Second-order partial derivatives

$f''_{xx}(x, y)$ or $\frac{d^2f}{dx\,dx}$ or $\frac{d^2f}{dx^2}$

$f''_{y}(x, y)$ or $\frac{d(f)}{dx}$

$f''_{yx}(x, y)$ or $\frac{d^2f}{dy\,dx}$

$f(x, y)$

$f'_{y}(x, y)$ or $\frac{d(f)}{dy}$ $\quad$ $\mathrm{f}''_{\mathrm{yx}}(x, y)$ or $\frac{d^2f}{dx\,dy}$

$f''_{yy}(x, y)$ or $\frac{d^2f}{dy\,dy}$ or $\frac{d^2f}{dy^2}$

The second-order partial derivatives f''_{xx} and f''_{xy} are obtained by differentiating f''_{x}, with respect to x and with respect to y respectively. Similarly the second-order partial derivatives f''_{yx} and f''_{yy} are obtained by differentiating *fy*, with respect to x and with respect to y respectively.

The cross (or mixed) partial derivative f''_{xy} or f''_{yx} indicates that first the primitive function has been partially differentiated with respect to one independent variable and then that partial derivative has in turn been partially differentiated with respect to the other independent variable:

$$f''_{xy} = \left(f'_x\right)'_y = \frac{d}{dy}\left(\frac{dz}{dx}\right) = \frac{d^2z}{dydx}$$

$$f''_{yz} = \left(f'_y\right)'_x = \frac{d}{dx}\left(\frac{dz}{dy}\right) = \frac{d^2z}{dxdy}$$

The cross partial is a second-order derivative, which are equal always. That is

$$\frac{d^2z}{dydx} = \frac{d^2z}{dxdy}$$

$$or, f''_{xy} = f''_{yx}$$

This is illustrated by the following examples.

Example–5:

Determine (a) first, (b) second and (c) cross partial derivatives of $Z = 7x^3 + 9xy + 2y^5$.

Solution:

$$\text{(a) } \frac{dz}{dx} = Z_x = 21x^2 + 9y; \qquad \frac{dz}{dy} = Z_y = 9x + 10y^4.$$

$$\text{(b) } \frac{d^2z}{dx^2} = Z_{xx} = 42x; \qquad \frac{d^2z}{dy^2} = Z_{yy} = 40y^3.$$

$$\text{(c) } \frac{d^2z}{dydx} = \frac{d}{dy} \cdot \frac{dz}{dx} = \frac{d}{dy}(21x^2+9y) = Z_{xy} = 9$$

$$\frac{d^2z}{dxdy} = \frac{d}{dx} \cdot \frac{dz}{dy} = \frac{d}{dx}(9x+10y^4) = Z_{yx} = 9.$$

Example–6:

Determine all first and second-order derivatives of

$Z = (x^2+3y^3)^4$

Solution:

$$\frac{dz}{dx} = 4\,(x^2+3y^3)^3 \cdot 2x = 8x\,(x^2+3y^3)^3$$

$$\frac{dz}{dy} = 4\,(x^2+3y^3)^3 .\ 9y^2 = 36y^2\,(x^2+3y^3)^3$$

$$\frac{d^2z}{dx^2} = 8x[3(x^2+3x^3)^2\,(2x)] + (x^2+3y^3)^3 \cdot 8$$

$$= 48x^2\,[x^2+3x^3]^2 + 8(x^2+3y^3)^3$$

$$\frac{d^2z}{dy^2} = 36y^2\,[3(x^2+3y^3)^2\,(9y^2)] + (x^2+3y^3)^3\ 72y$$

$$\frac{d^2z}{dydx} = \frac{d}{dy} \cdot \frac{dz}{dx} = 8x\,[3(x^2+3y^3)^2\,(9y^2)]+0$$

$$= 216xy^2\,(x^2+3y^3)^2$$

$$\frac{d^2z}{dxdy} = \frac{d}{dx} \cdot \frac{dz}{dy} = 36y^2\,[3\,(x^2+3y^3)^2 .\ 2x]$$

$$= 216xy^2\,(x^2+3y^3)^2$$

Optimization of Multivariate Functions

For a multivariate function such as $z = f(x, y)$ to be at a relative minimum or maximum, the following three conditions must be fullfiled /met:

1. Given $z = f(x, y)$, determine the first-order partial derivatives, $f'_x(x, y)$ and $f'_y(x, y)$ and all critical point/values (a, b); that is, determine all values (a, b) such that $f'_x(a, b) = f''_y(a, b) = 0$

2. Determine the second-order partial derivatives, $f''_{xx}(x, y), f''_{xy}(x, y), f''_{yx}(x, y)$ and $f''_{yy}(x, y)$

[Note: the cross partial derivatives $f''_{xy}(x, y)$ (x, y) and $f''_y(x, y)$ (x, y) must be equal to one another; otherwise the function is not continuous.]

3. Where (a, b) is a critical point on f, let D = f''_{xx} (a, b). $f''_{yy}(a, b) - [f''_{xy}(a, b)]^2$

   Then

   (i) If D>0 and $f''_{xx}(a, b) < 0$, f has a relative maximum at (a, b)

   (ii) If D>0 and $f''_{xx}(a, b) > 0$, f has a relative minimum at (a, b).

   (iii) If D<0, f has neither a relative maximum nor a relative minimum at (a, b)

   (iv) If D = 0, no conclusion can be drawn; further analysis is required.

This is illustrated by the following example.

Example–7:

Determine the critical points and specify whether the function had a relative maximum or minimum,

$$z = 2y^3 - x^3 + 147x - 54y + 12.$$

Solution:

By taking the first-order partial derivatives, setting them equal to zero, and solving for x and y:

$z_x = -3x^2 + 147 = 0$ $\qquad$ $z_y = 6y^2 - 54 = 0$

or, $x^2 = 49$ $\qquad$ or, $6y^2 = 54$

$\therefore x = \pm 7$ $\qquad$ $\therefore y = \pm 3$

This mean that we must investigate four critical points, namely (7, 3), (7, –3), (–7, 3) and (–7, –3).

The second-order partial derivatives are

$Z_{xx} = -6x$ $Z_{yy} = 12y$

(1) Z_{xx} (7, 3) = –6(7) = –42 <0 Z_{yy} (7, 3) = 12 × 3 = 36 > 0

(2) Z_{xx} (7, –3) = –6 (7) = –42 <0 Z_{yy} (7, –3) = 12*x* –3 =–26 < 0

(3) Z_{xx} (–7, 3) = –6(–7) = 42>0 Z_{yy} (–7,3) = 12*x*3 = 36 > 0

(4) Z_{xx} (–7, –3) = –6 (–7) = 42>0 Z_{yy} (–7, –3) = 12*x* –3 = –36 < 0

Since there are different signs for each of the second-order partials in (1) and (4), the function cannot be at a relative maximum or minimum at (7, 3) or (–7, –3). When f''_{xy} and f''_{yy} are of different signs, $\left(f''_{xx}.f''_{yy}\right)$cannot be greater than f''_{xy} and the function is at a saddle point.

With both signs of second-order partials negative in (2) and positive in (3), the function may be at a relative maximum at (7, –3) and at a relative minimum at (–7, 3), but the third condition must be tested first to ensure against the possibility of an inflection point.

From the first partial derivative, we obtain cross partial derivatives and check to make sure that Z_{xx} *(a, b),* Z_{yy} *(a, b) > [*Z_{xy} *(a, b)]*^{2}

Hence, $Z_{xy} = 0$ and $Z_{yx} = 0$

$Z_{xx}(a, b).Z_{yy}(a, b) > [Z_{xy}(a, b)]^2$

From (2), (–42). (–36)>$(0)^2$

□□∴1512 > 0

From (3), (42). (36) > $(0)^2$

∴ 1512 > 0.

Hence the function has a relative maximum at (7, –3) and a relative minimum at (–7, 3).

Questions for Review

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. Determine first, second and cross – partial derivatives of

   $f(x, y) = 2x^2 + 4xy^2 - 5y^2 + y^3$

2. Determine the first and second–order partial derivatives of the function

   $f(x, y, z) = x^2e^y$ In z.

3. Examine the function, $z(x, y) = x^2 + y^2 - 4x + 6y$ for relative maxima or minima by using second- order derivative test.

4. Find the partial derivatives for $z = (6x + 4)(4x + 2y)$

Lesson-3: Optimization with Lagrangian Multipliers and Cobb-Douglas Production Functions

After studying this lesson, you should be able to:

- Discuss the nature of constrained optimization with lagrangian multipliers;
- Discuss the nature of optimization of Cobb-Douglas Production functions;
- Apply the techniques to solve the relevant problems.

Constrained Optimization with Lagrangian Multipliers

Differential calculus is also used to maximize or minimize a function subject to constraints. Given a function $f(x, y)$ subject to a constraint g $(x, y) = k$ (a constant), a new function F can be formed by– (1) setting the constraint equal to zero, (2) multiplying it by λ (the lagrange multiplier), and (3) adding the product to the original function :

Differential calculus is also used to maximize or minimize a function subject to constraints

$$F(x, y, \lambda) = f(x, y) + \lambda\,[k - gcx, y)].$$

Here $F(x, y, \lambda)$ = Lagrangian functions

$f(x, y)$ = original or objective function

and $g(x, y)$ = constraint.

Since the constraint is always set equal to zero, the product $\lambda\,[k - g(x, y)]$ also equals zero, and the addition of the term does not change the value of the objective function. Critical values x_0, y_0 and λ_0, at which the function is optimised, are found by taking the partial derivatives of F with respect to all three independent variables, setting them equal to zero, and solving simultaneously:

Since the constraint always set equal to zero, the product λ [k–g (x, y)] also equals zero.

$$Fx(x, y, \lambda) = 0 \qquad Fy(x, y, \lambda) = 0; \qquad Fz(x, y, \lambda) = 0$$

This is illustrated by the following examples.

Example–1:

Determine the critical points and optimise the function
$Z = 4x^2 + 3xy + 6y^2$ subject to the constraint $x + y = 56$.

Solution:

Setting the constraint equals to zero, $56 - x - y = 0$

The lagrangian expression is,

$$F(x, y, \lambda) = Z = 4x^2 + 3xy + 6y^2 + \lambda(56 - x - y) \quad ...(1)$$

and the partial derivatives are

$$\frac{d(F)}{dx} = Z_x = 8x + 3y - \lambda = 0 \qquad \text{... (2)}$$

$$\frac{d(F)}{dy} = Z_y = 3x + 12y - \lambda = 0 \qquad \text{... (3)}$$

$$\frac{d(F)}{dl} = Z\lambda = 56 - x - y = 0 \qquad \text{... (4)}$$

Subtracting (3) from (2) to eliminate λ gives

$5x - 9y = 0$

or $5x = 9y$

$\therefore x = 1.8y$ ... (5)

Substituting $x = 1.8y$ in equation (4)

$1.8y + y - 56 = 0$

or, $2.8y = 56$

$\therefore\ y = {}^{56}/_{2.8} = 20.$

Substituting $y = 20$ in equation (5) we get

$x = (1.8 \times 20) = 36$

Substituting $x = 36$ and $y = 20$ in equation (2) we have

$8\ (36) + 3(20) - \lambda = 0$

or, $288 + 60 - \lambda = 0$

or, $348 - \lambda = 0$

$\therefore\ \lambda = 348$

Substituting the critical values, $x = 36$, $y = 20$ and $\lambda = 348$ in lagrangian function, we have,

$Z = 4(36)^2 + 3(36)\ (20) + 6(20)^2 + 348(56 - 36 - 20)$

$\quad = 4\ (1296) + 3(720) + 6(400) + 348\ (0) = 9744.$

Example–2:

Use lagrange multipliers to optimize the function, $f\ (x,\ y) = 26x–3x^2 + 5xy - 6y^2 + 12y$ subject to the constraint $3x + y = 170$.

Solution:

The lagrangian function is

$$F = 26x–3x^2+5xy–6y^2+12y+\lambda\ (170–3x–y) \qquad \text{... (1)}$$

Thus $F_x = 26–6x + 5y – 3\lambda = 0$ ... (2)

$F_y = 5x–12y +12–\lambda = 0$ ... (3)

$F_\lambda = 170 – 3x – y = 0$ ... (4)

Multiplying equation (3) by 3 and subtracting from equation (2) to eliminate λ, we have

$–21x + 41y – 10 = 0$ ... (5)

Multiplying equation (4) by 7 and subtracting from equation (5) to eliminate *x,* we have

$48y – 1200 = 0$

$\therefore\ 48y = 1200$

$\therefore\ y = 25$

Substituting y=25 in equation (4), we get

$170 – 3x – 25 = 0$

or, $–3x = –145$

$\therefore\ x = 48\frac{1}{3}$

Then substituting $x = 48\frac{1}{3}$ and $y = 25$ in equation (2), we get $\lambda = – 46\frac{1}{3}$

Using x0 = $48\frac{1}{3}$, $y = 25$ and $\lambda = – 46\frac{1}{3}$ in lagrangian expression, we get

$$F=26\left(48\frac{1}{3}\right)–3\left(48\frac{1}{3}\right)^2+5\left(48\frac{1}{3}\right)(25)–6(25)^2+12(25)+\left(170–3\left(48\frac{1}{3}\right)–25\right)$$

$\therefore\ F = – 3160$

Example–3:

Determine the critical points and the constrained optima for $Z = x^2 + 3xy + y^2$ subject to $x + y = 100$

Solution:

The lagrangian expression is $F(x, y, \lambda) = f(x, y) + \lambda.\ g(x, y)$

$= x^2 + 3xy + y^2+\lambda\ (x + y –100)$

and the partial derivatives are

$$\frac{dF}{dx} = 2x + 3y + \lambda = 0$$

$$\frac{dF}{dy} = 3x + 2y + \lambda = 0$$

$$\frac{dF}{d\lambda} = x + y - 100 = 0$$

The three equations are solved simultaneously to obtain $x = 50$, $y = 50$ and $\lambda = 250$.

The critical values of x, y and λ can be substituted into the lagrangian expression to obtain the constrained optimum.

$$\text{F}\ (50, 50, -250) = (50)^2 + 3(50)\ (50) + (50)^2 - 250\ (50 + 50 - 100)$$

$$= 2500 + 7500 + 2500 - 0 = 12{,}500.$$

To determine whether the function reaches a maximum or a minimum, we evaluate the function at points adjacent to $x = 50$ and $y = 50$. The function is a constrained maximum, since adding Δx and Δy to the function in both directions gives a functional value less than the constrained optimum, i.e.

$$F\ (49, 51, -250) = 12499$$

$$F\ (51, 49, -250) = 12{,}499$$

Optimization of Cobb-Douglas Production Functions

Economic analysis is frequently couched in terms of the Cobb-Douglas production function, $Q = AK^{\alpha}L^{\beta}$ (A>0; 0<α, β<1) where Q is the quantity of output in physical units, K the quantity of capital, and L the quantity of labor. Here α (the output elasticity of capital) measures the percentage change in Q for a 1 percent change in K while L is held constant; β (the output elasticity of labor) is exactly parallel; and A is an efficiency parameter reflecting the level of technology.

rict cobb-douglas tion, in which α + 1, shows constant rns to scale and reasing returns to e if α+β <1.

A strict cobb-douglas function, in which $\alpha + \beta = 1$, shows constant returns to scale and decreasing returns to scale if $\alpha+\beta$ <1. A cobb-douglas function is optimized subject to a budget constraint. This is illustrated by the following examples.

Example–4:

Optimize the Cobb-Douglas production function, $Q = \text{k}^{0.4}\ \text{L}^{0.5}$, given $P_k = 3$, $P_L = 4$ and $B = 108$.

Solution:

By setting up the lagrangian function, we get

$$Q = k^{0.4}\ L^{0.5}\ \lambda\ (108 - 3k - 4L)$$

Using the simple power function rule, taking the first-order partial derivatives, setting them equal to zero and solving simultaneously for K_0 and L_0 (and λ_0, if desired)

$$\frac{d(Q)}{dk} = Q_k = 0.4K^{-0.6.}\, L^{0.5} - 3\lambda = 0 \qquad \text{... (1)}$$

$$\frac{dQ}{dL} = Q_L = 0.5K^{0.4}.\, L^{-0.5} - 4\lambda = 0 \qquad \text{... (2)}$$

$$\frac{dQ}{dl} = 108 - 3k - 4L = 0 \qquad \text{... (3)}$$

Rearranging, then dividing (1) by (2) to eliminate λ, we get.

$$\frac{0.4K^{-0.6}.L^{05}}{0.5K^{0.4}.L^{-0.5}}$$

or, $.8k^{-1}L^{1} = 0.75$

or, $\frac{L}{K} = \frac{0.75}{0.8}$

$\therefore L = 0.9375k.$

Substituting L = 0.9375k in equation (3), we get

$108 - 3k - 4\,(0.9375k) = 0$

$\therefore K_o = 16.$

Then by substituting $K_o = 16$ in equation (3), we have

$108 - 3\,(16) - 4L = 0$

or, $108 - 48 - 4L = 0$

or, $-4L = -60$

$\therefore L_0 = 15$

Example–5:

From the following information find least cost input combination and the minimum cost of production:

$Q_n = 500$, P_d = Tk.10 and P_k = Tk.0.50. Subject to $Q_n = 1.01\, L^{0.75}.\, K.^{0.25}$

Solution:

It can be stated that minimize,

$C = 10L + 0.50K$

Subject to $500 = 1.01L^{0.75}\, K^{0.25}$

Where, A. b, a are positive constants, a +b=1

b = 0.75; a = 0.25; A = 1.01

This can be solved through the lagrangian multiplier technique. The lagrangian expression would be

$V = LP_L + KP_K + \lambda\,[Q - AL^b K^a]$

or, $V = 10L + 0.50K + \lambda\,[500 - 1.01\,L^{0.75}\,K^{0.25}]$

Where λ is the lagrangian multiplier, and V is the minimum cost.

The necessary condition for optimization is that each of the partial derivatives of V with respect to L And K must be equal to zero.

$\frac{dV}{dL}$ or $V'(L) = 10 - \lambda\,[1.01(0.75)\,L^{-0.25}\,K^{0.25}] = 0$

So, $\lambda = \frac{10}{0.7575\,L^{-0.25}.K^{0.25}}$ ----------- (L)

$\frac{d(V)}{dK}$ or $V'(K) = 0.50 - \lambda\,[1.01\,L^{0.75}(0.25)K^{-0.75}] = 0$

$\therefore\ 0.50 - \lambda\,[0.2525\,L^{0.75}\,K^{-0.75}] = 0$

So, $\lambda = \frac{0.50}{0.2525\,L^{0.75}.K^{-0.75}}$ ---- -----(K)

We know that

$$\frac{\text{Marginal physical product of labor (MPPL)}}{\text{Marginal physical product of capital (MPPK)}} = \frac{P_L}{P_K}$$

Then we get,

$$\frac{0.7575\,L^{-0.25}\,K^{0.25}}{0.2525\,L^{0.75}\,K^{-0.75}} = \frac{10}{0.50}$$

or, $\frac{3K}{L} = \frac{10}{0.50}$

or, 1.5K = 10L

So, $K = \frac{10L}{1.5}$ $\therefore K = 6.67L$

Substituting the value of K in the original subject to the production function, we have

$500 = 1.01\ L^{0.75}\,(6.67L)^{0.25}$

Taking logarithm of both sides, we get

log 500 = log1.01+0.75logL+0.25log(6.67)+0.25log L

or, log 500 – log 1.01 = log L + 0.25log(6.67)

or, 2.69897 – 0.00432 = log L + 0.25(0.82413)

or, log L = 2.48862

So, L = anti-log 2.48862 = 308.05 = 309 units (app),

Substituting the value of L, we have

K = 6.67 L

= 6.6 (308.7) = 2054.69 units (app) rounded to = 2055 Units

Substituting the value of L and K, we have,

C = 10L + 0.50K = 10 (309) + 0.50 (2055) = 3090 + 1027.50

= Tk.4117.50

Hence the least cost input combination is L = 309 and K = 2055 units and the minimum cost of production is Tk.4117.50

Questions for Review

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. Defermine the critical points and optimize the function $f(x, y)=5x^2+6xy-3y^2+10$ subject to the constraint $x+2y=24$.

2. Find the maxima and minima of the following function subject to the constraining equation.

   $f(x, y)=12xy - 3y^2 - x^2$ and $x + y = 16$.

3. Optimize the following Cobb-Douglas production functions subject to the given constraint. Q = K0.3L0.5 subject to 6K + 2L = 384.

4. Optimize the following Cobb-Douglas production function subject to given constraints by $Q = 10k^{0.7} \times L^{0.1}$ given $P_k = 28$, $P_L = 10$ and $\beta = 4000$.

Lesson-4: Business Applications of Differentiation

Lesson Objectives:

After studying this lesson, you should be able to:

- State the key concepts related to business applications of differentiation;
- Apply the techniques of differentiation to solve business problems.

Key Concepts

Total costs (TC): Total cost is the combination of fixed cost and variable cost of output. If the production increases, only total variable cost will increase in direct proportion but the fixed cost will remain unchanged within a relevant range.

Total revenue (TR): Total revenue is the product of price/demand function and output.

Profit: Profits are defined as the excess of total revenue over total costs. Symbolically it can be expressed as, P (profit) = TR – TC. i.e., (Total Revenue – Total Cost)

The rules for finding a maximum point tell us that P is maximized when the derivative of the profit function is equal to zero and the second derivative is negative

The rules for finding a maximum point tell us that P is maximized when the derivative of the profit function is equal to zero and the second derivative is negative. If we denote the derivatives of the revenue and cost functions by dTR and dTC we have,

'P' is at a maximum when dTR - dTC = 0. This equation may be written as dTR = dTC.

The derivative of the total revenue function must be equal to the derivative of the total cost function for profits to be maximized.

Profit Maximizing output = $\frac{d(\text{profit function})}{dx}$

Condition: In case of maximization, the conditions are $\frac{dp}{dx} = 0$ and $\frac{d^2p}{dx^2}$ must be Negative.

Cost Minimizing output = $\frac{d(\text{total cost function})}{dx}$

Condition: In case of minimization, the conditions are $\frac{dTCy}{dx} = 0$ and $\frac{d^2{}_{TC}}{dx^2}$ must be Positive.

Marginal Cost (MC): MC is the extra cost for producing one additional unit when the total cost at certain level of output is known. Hence, it is the rate of change in total cost with respect to the level of output at the point where total cost is known. Therefore, we have, MC = $\frac{dTC}{dx}$ where total cost (TC) is a function of x, the level of output.

Marginal Production (MP): MP is the incre–mental production, i.e., the additional production added to the total production, i.e. MP = $\frac{dTP}{dx}$

Marginal Revenue (MR): MR is defined as the change in the total revenue for the sale of an extra unit. Hence, it is the rate of change total in revenue with respect to the quantity demanded at the point where total revenue is known. Therefore, we have, MR = $\frac{dTR}{dx}$ where total revenue (TR) is a function of x, the quantity demanded.

MR is defined as the change in the total revenue for the sale of an extra unit

Let us illustrate these concepts by the following examples.

Example–1:

The profit function of a company can be represented by P =f (x) = x – $0.00001x^2$, where x is units sold. Find the optimal sales volume and the amount of profit to be expected at that volume.

Solution:

The necessary condition for the optimal sales volume is that the first derivative of the profit function is equal to zero and the second derivative must be negative. Where the profit function is:

$P = x - 0.00001x^2$

$$\frac{dP}{dx} = \frac{d}{dx}\left(x - 0.00001x^2\right)$$

Marginal Profit = 1 – 0.00002x

To get maximum profit now we put marginal profit = 0

So, 1 – 0.00002x = 0

or, 0.00002x = 1

So, x = 0.00002 = 50,000 units.

The second derivative of profit function, i.e.

$$\frac{d^2P}{dx^2} = \frac{d}{dx}(1 - 0.00002x) = -0.00002 < 0$$

Now by putting the value of x in profit function we get maximum profit.

$P = x - 0.00001x^2$

$= 50,000 - 0.00001(50,000)^2 = 50,000 - 0.00001(2500000000)$

= 50,000 – 25,000 = 25,000.

The optimum output for the company will be 50,000 units of x and maximum profit at that volume will be Tk.25,000.

Example – 2:

If the total manufacturing cost 'y' (in Tk.) of making x units of a product is : y = 20x + 5000, (a) What is the variable cost per unit? (b) What is the fixed cost? (c) What is the total cost of manufacturing 4000 units? (d) What is the marginal cost of producing 2000 units?

Solution:

We have the cost-output equation : y = 20x + 5000. We know that, if the production increases, only total variable cost will increase in direct proportion but the fixed cost will remain unchanged in total. So, the derivative of y with respect to the increase in x by 1 unit will give the variable cost per unit.

(a) Variable cost per unit = $\frac{d}{dx}$ (cost-output equation)

$$= \frac{d}{dx}(y)$$

$$= \frac{d}{dx}(20x + 5000) = 20.$$

∴ Variable cost per unit is Tk.20.

(b) Total fixed cost will remain unchanged even if we don't produce any unit. If we don't produce any unit, there will be no variable cost and only fixed cost will be the total cost. So, if we put x=0 in the cost-output equation, we will get the fixed cost.

∴ Fixed Cost = y = [20.(0) + 5,000] = Tk.5000

(c) If we put x=4000 in the cost-output equation, we will get the total cost of producing 4,000 units.

∴ Total cost of producing 4000 units = y = 20 (4,000) + 5,000 = Tk.85,000.

(d) We know that the marginal cost of 'n'th unit = $TC_n - TC_{n-1}$

∴ Marginal cost of 2000th unit

= TC of 2000 units – TC of (2000–1) units

= [20 (2000) + 5000] – [20 (1999 Tk. + 5000] = [45,000 – 44,980] = 20.

Example–3:

The total cost of producing x articles is $\frac{5}{4}\ x^2 + 175x + 125$ and the price at which each article can be sold is $250 - \frac{5}{4}x$. What should be the output for a maximum profit. Calculate the profit.

Solution:

$$\text{Total revenue (TR)} = \left[\left(250 - \frac{5}{4}x\right)x\right] = 250 - \frac{5x^2}{2}$$

$$\text{Total cost (TC)} = \frac{5x^2}{4} + 175x + 125$$

So, profit (P) = TR – TC

$$= 250x - \frac{5x^2}{2} - \frac{5x^2}{4} - 175x - 125 = \frac{-10x^2 - 5x^2}{4} + 75x - 125 = \frac{-15x^2}{4} + 75x - 125$$

$$\frac{d(p)}{dx} = \frac{-30x}{4} + 75$$

The necessary condition for optimization is that the first derivative of a profit function is equal to zero and second derivative must be negative. According to the assumption, we have

$$\frac{-30x}{4} + 75 = 0$$

or, $\frac{-30x + 300}{4} = 0$

or, $-30x + 300 = 0$

or, $-30x = -300$

or, $x = 10$

and $\frac{d^2p}{dx_2} = \frac{d}{dx} . \frac{dp}{dx} = \frac{d}{dx}\left(\frac{-30x}{4} + 75\right) = \frac{-30}{4} < 0$

Therefore the profit is maximum when the output (x) is 10

$$\text{Profit function} = \frac{-15x^2}{4} + 75(x) - 125$$

Putting the value of x in profit function, we get,

$$\text{Profit} = \frac{-15(10)^2}{4} + 75(10) - 125$$

$$= \frac{-1500}{4} + 750 - 125 = -375 + 750 - 125 = 750 - 500 = 250$$

Hence the profit is Tk. 250.

Example–4:

The total cost function of a firm is $C = \frac{1}{3}x^3 - 5x^2 + 28x + 10$, where C is total cost and x is output. A tax at the rate of Tk.2 per unit of output is imposed and the producer adds it to his cost. If the market demand function is given by $P = 2530 - 5x$, where P is the price per unit of output, find the profit maximizing output and price.

Solution:

Total Revenue (TR) = $(2530 - 5x)\, x = 2530x - 5x^2$

Total cost (TC) = $(\frac{1}{3}x^3 - 5x^2 + 28x + 10)$ + (Taxes i.e.$2x$)

$$= \frac{1}{3}x^3 - 5x^2 + 28x + 10 + 2x = \frac{1}{3}x^3 - 5x^2 + 30x + 10$$

So, profit (P) = TR – TC

$= 2530x - 5x^2 - \frac{1}{3}x^3 + 5x^2 - 30x - 10 = 2500x - \frac{1}{3}x^3 - 10.$

So, $\frac{d(P)}{dx} = 2500 - \frac{3x^2}{3} = 2500 - x^2$

The necessary condition for maximization is that the first derivative of a profit function is equal to zero and second derivative must be negative.

According to the condition, we can write

$2500 - x^2 = 0$

or, $-x^2 = -2500$

So, $x = 50$

and $\frac{d_2 p}{dx^2} = -2x = -2x\ 50 = -100 < 0$

Hence the profit maximizing output of the firm is 50 units.

At this level, the price is given by

Price = $2530 - 5x = 2530 - 5 \times 50 = 2530 - 250$ = Tk.2280.

Example–5:

A motorist has to pay an annual road tax of \$50 and \$110 for insurance. His car does 30 miles to the gallon which costs 75 Pence (per gallon). The car is serviced every 3000 miles at a cost of \$20, and depreciation is calculated in pence by multiplying the square of the mileage by 0.001.

Obtain an expression for the total annual cost. Hence find an expression for the average total cost per mile and calculate the annual mileage which will minimize the average cost per mile.

Solution:

Suppose he covers x miles in a year.

Tax per annum = \$50; Insurance per annum \$ 110

Cost of petrol $= \frac{.75x}{30}$; Service charges $= \frac{20x}{3000}$ .20

Depreciation $= \frac{0.001x^2}{100}$ (0.001 is in pence and is divided by 100 to get \$ amount)

Total cost: $C = 50+110\frac{0.75x}{30}+\frac{20x}{3000}+0.00001x^2$

Average TC per mile: $\frac{C}{x} = M = \frac{160}{x} + \frac{0.75}{30} + \frac{20}{3000} + 0.00001x$

$\frac{dM}{dx} = \frac{d}{dx}\ 160\ x^{-1} + \frac{0.75}{30} + \frac{20}{3000} + 0.00001x$

$= -160\ x^{-2} + 0 + 0 + 0.00001$

$= -\frac{160}{x^2} + 0.00001$

The necessary condition for minimization of cost is $\frac{dM}{dx} = 0$, and $\frac{d^2M}{dx^2}$ must be positive.

According to the condition, we can write $\frac{-160}{x^2} + 0.00001 = 0$

or, $\frac{160}{x^2} = 0.00001$

or, $0.00001x^2 = 160$

or, $x^2 = \frac{160}{0.00001} = 16000000$

or, $x = 4000$

The average cost is a minimum since $\frac{d^2M}{dx^2} = -160\ x^{-2} + 0.00001$

$= 320\ x^{-3} + 0 = \frac{320}{x^3} > 0$

So, the motorist can cover 4000 miles in a year to minimize the average cost per mile.

Example–6:

The yearly profits of ABC company are dependent upon the number of workers (x) and the number of units of advertising (y), according to the function

$$P(x,y) = 412x + 806y - x^2 - 4y^2 - xy - 50{,}000$$

(i) Determine the number of workers and the number of units in advertising that results in maximum profit.

(ii) Determine maximum profit.

Solution:

(i) To determine the values of x and y, we equate the partial derivatives of the profit function with zero.

$$P_x(x, y) = 412 - 2x - y = 0 \quad ... (1)$$

$$P_y(x, y) = 806 - x - 8y = 0 \quad ... (2)$$

The two equation are solved simultaneously to obtain the values of x and y.

$$412 - 2x - y = 0$$

$$1612 - 2x - 16y = 0$$

$$-1200 + 15y = 0$$

$$\text{or, } 15y = 1200$$

$$\therefore y = 80$$

Substituting the value of y in equation (1), we get

$$412 - 2x - 80 = 0$$

$$\text{or } -2x = -332$$

$$\therefore x = 166$$

and

$$P_{xx}(x, y) = -2 < 0$$

$$P_{yx}(x, y) = -8 < 0$$

(ii) The profit generated from using these values is:

$$P(166, 80) = 412(166) + 806(80) - (166)^2 - 4(80)^2 - (166)(80) - 50000$$

$= 68392 + 64640 – 27556 –25600 – 13280– 50,000$

$=16595$

Example – 7:

The cost of construction (c) of a project depends upon the number of skilled workers (x) and unskilled workers (y). If cost is given by, C (x, y) = $50000+9x^3–72xy+9y^2$

(i) Determine the number of skilled workers and unskilled workers that results in minimum cost.

(ii) Determine the minimum cost.

Solution:

To determine the number of skilled workers (x) and unskilled workers (y), we equate the partial derivatives of the cost function with zero.

$C_x (x, y) = 27x^2–72y = 0$ ... (1)

$C_y (x, y) = –72x + 18y = 0$ ... (2)

Solving the two equations simultaneously gives

$27x^2 – 72y = 0$

$–288x + 72y = 0$ [Multiplying equation (2) by 4]

By adding $27x^2 – 288x = 0$

or, $x (27x – 288) = 0$

or, $27x – 288 = 0$

or, $27x = 288$

$\therefore x = 10.66$ rounded to

Substituting $x = 11$ in equation (2) we get

$–72 (11) + 18y = 0$

or, $–792 +18y = 0$

or, $18y = 792$

$\therefore y = \frac{792}{18} = 44$

(ii) Putting the value of x and y in cost function, we get:

$C (11, 44) = 50000 + 9 (11)^3–72 (11) (44) + 9 (44)^2$

$= 44,555$

Questions for Review:

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. A study has shown that the cost of producing sign pens of a manufacturing concern is given by, $C = 30 + 1.5x + 0.0008x^2$. What is the marginal cost at x = 1000 units? If the pens sells for Tk.5.00 each, for what values of x does marginal cost equal marginal revenue?

2. The demand function faced by a firm is $P=500 - 0.2x$ and its cost function is $C=25x+10,000$. Find the optimal output at which the profits of the firm and maximum. Also find the price it will charge.

3. The demand function of a profit maximizing monopolist is, $P-3Q-30 = 0$ and his cost function is, $C = 2Q^2+10Q$. If a tax of taka 5 per unit of quantity produced is imposed on the monopolist, calculate the maximum tax revenue obtained by the Government.

4. A company produces two products, x units of type –A and y units of type –B per month. If the revenue and cost equation for the month are given by-

   $R (x, y) = 11x +14y, C (x, y) = x^2 - xy +2y^2 + 3x + 4y +10$

5. Total cost function is given by, $TC = 3Q^2+7Q+12$, Where C=Cost of production, Q = output. Find

   (i) marginal cost

   (ii) Average cost if Q = 50

6. The total cost of production for the electronic module manufactured by ABC Electronics is

   $TC = 0.04Q^3 - 0.30Q^2 + 2Q + 1$

7. Determine MC, AC, AFC and TVC function. Find out the output level for which MC is minimum. What is the amount of MC, AC and TC at this level of output.

8. The transport authority of the city corporation areas has experimented with the fare structure for the city's public bus system. The new system is fixed fare system in which a passenger may travel between two points in the city for the same fare. From the survey results, system analysis have determined an appropriate demand function, P=2000–125Q, where Q equals to the average number of riders per hours and p equals the fare in taka.

   Required:

(i) Determine the fare which should be charged in order to maximize hourly bus for revenue.

(ii) How many rider are expected per hour under this fare?

(iii) What is the expected maximum annual revenue?

Maxima and Minima

9

This unit is designed to introduce the learners to the basic concepts associated with Optimization. The readers will learn about different types of functions that are closely related to optimization problems. This unit discusses maxima and minima of simple polynomial functions and develops the concept of critical points along with the first derivative test and next the concavity test. This unit also discusses the procedure of determining the optimum (maximum or minimum) point with single variable function, multivariate function and constrained equations. Some relevant business and economic applications of maxima and minima are also provided in this unit for clear understanding to the learners.

Blank Page

Lesson-1: Optimization of Single Variable Function

After studying this lesson, you should be able to:

- Describe the concept of different types of functions;
- Explain the maximum, minimum and point of inflection of a function;
- Describe the methodology for determining optimization conditions for mathematical functions with single variable;
- Determine the maximum and minimum of a function with single variable;
- Determine the inflection point of a function with single variable.

Introduction

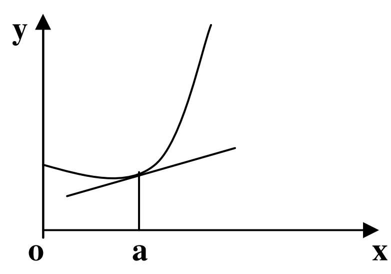
Optimization is a predominant theme in management and economic analysis. For this reason, the classical calculus methods of finding free and constrained extrema and the more recent techniques of mathematical programming occupy an important place in management and economics. The most common criterion of choice among alternatives in economics is the goal of maximizing something (i.e. profit maximizing, utility maximizing, growth rate maximizing etc.) or of minimizing something (i.e. cost minimizing). Economically, we may categorize such maximization and minimization problems under the general heading of optimization, which means 'the quest for the best'. The present lesson is devoted to a brief discussion of optimization with single variable function.

The most common criterion of choice among alternatives in economics is the goal of maximizing something or of minimizing something.

Increasing Function

A function $f(x)$ is said to be increasing at $x = a$ if the immediate vicinity of the point $(a, f(a))$ the graph of the function rises as it moves from left to right.

Since the first derivative measures the rate of change and slope of a function, a positive first derivative at $x = a$ indicates the function is increasing at x=a, i.e. $f'(a) > 0$ means increasing function at $x = a$.

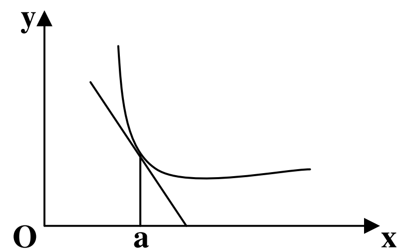


Decreasing Function:

A function $f(x)$ is said to be decreasing at $x = a$ if the immediate vicinity of the point $(a, f(a))$ the graph of the function falls as it moves from left to right. Since the first derivative measures the rate of change and slope

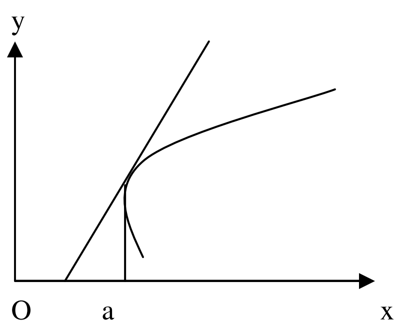
A function f (x) is said to be decreasing at x = a if the immediate vicinity of the point (a, f (a)) the graph of the function falls as it moves from left to right.

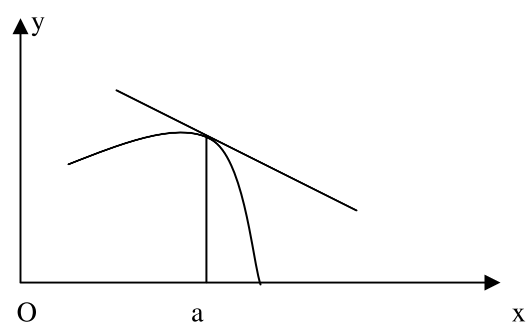
of a function, a negative first derivative at $x = a$ indicates the function is decreasing at $x = a$, i.e. $f'(a) < 0$; means decreasing function at $x = a$.



Concave Function:

A function $f(x)$ is concave at $x = a$ if in some small region close to the point $(a, f(a))$ the graph of the function lies completely below its tangent line. A negative second derivative at $x = a$ denotes the function is concave at $x = a$. The sign of the first derivative is immaterial for concavity.



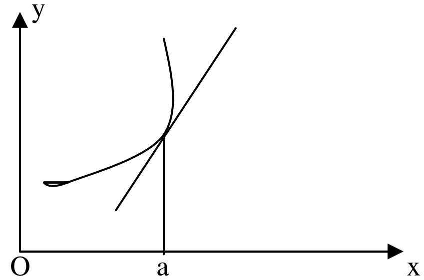


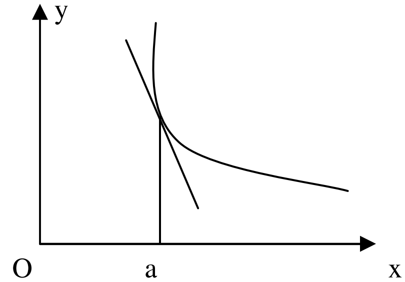
$f'(a) > 0;\ f''(a) < 0$ $\qquad$ $f'(a) < 0;\quad f''(a) < 0$

Convex Function:

A function f(a) is convex at x = a if in an area very close to the point (a, f(a)) the graph of the function lies completely above its tangent line.

A function $f(a)$ is convex at $x = a$ if in an area very close to the point $(a, f(a))$ the graph of the function lies completely above its tangent line. A positive second derivative at $x = a$ denotes the function is convex at $x = a$.



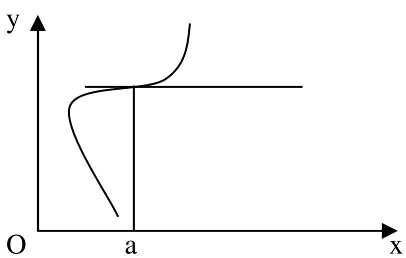


$f'(a) > 0;\ f''(a) > 0$ $\qquad$ $f'(a) < 0;\quad f''(a) > 0$

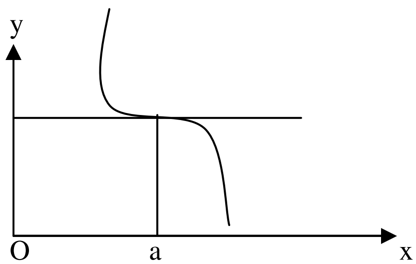
Inflection Points:

An inflection point is a point on the graph where the function crosses its tangent line and changes from concave to convex or vice versa. Inflection points occur only where the second derivative equals zero or is undefined. Hence, the sign of the first derivative is immaterial.

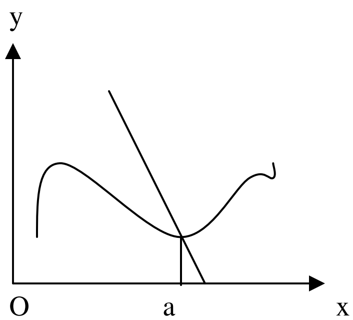
An inflection point is a point on the graph where the function crosses its tangent line and changes from concave to convex or vice versa.



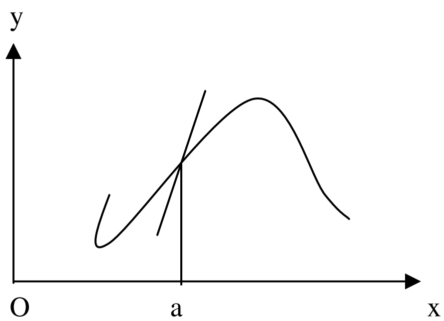
$f'(a) = 0; \; f''(a) = 0$



$f'(a) = 0; \quad f''(a) = 0$



$f'(a) < 0; \quad f''(a) = 0$



$f'(a) > 0; \quad f''(a) = 0$

Maxima:

A function *f (x)* is said to have attained at any of its maximum values at $x = a$ if the function ceases to increase and begins to decrease at $x = a$.

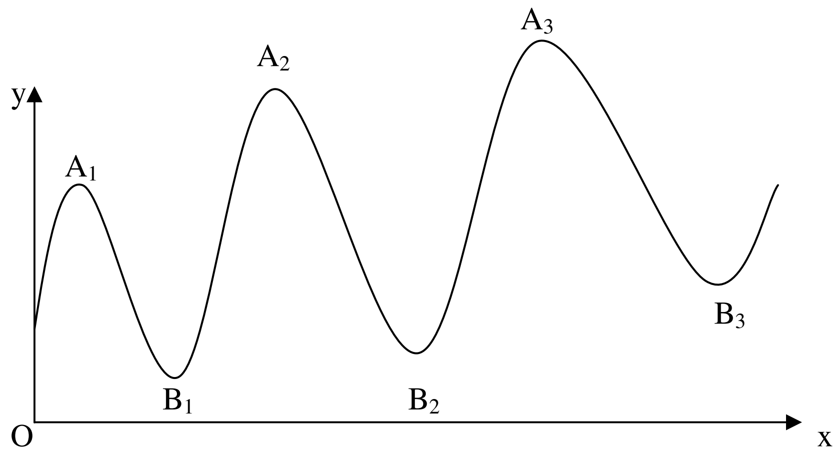
A function f (x) is said to have attained at any of its maximum values at x = a if the function ceases to increase and begins to decrease at x = a.

In other words, a function $f(x)$ is said to be maximum at a point $x = a$ if $f(a)$ is greater than any other values of $f(x)$ in the neighbourhood of $x = a$.

Minima:

A function $f(x)$ is said to have attained at any of its minimum values at $x = a$ if the function ceases to decrease and begins to increase at $x = a$.

In other words, a function $f(x)$ is said to be maximum at a point $x = a$ if $f(a)$ is less than any other values of $f(x)$ in the neighbourhood of $x = a$.



In the above figure, which represents graphically the function $y = f(x)$, a continuous function, has maximum values at A_1, A_2, A_3 and has minimum values at B_1, B_2, B_3.

From the figure, the following features regarding maxima and minima of a continuous function will be apparent:

(i) A function may have more than one maximum and minimum values. A function may have several maxima and minima in an interval where the function is defined.

(ii) It is not necessary that a maximum value of a function is always greater than other minimum values of the function. Any maximum value of a function may be less than any other minimum value of a function.

(iii) In between two maxima, there should be at least one minimum value of the function. Similarly, at least one maximum value of the function must lie between two minimum values of the function. In other words, there is a minimum value of the function between two consecutive maximum values and vice versa. Thus we observe that maximum and minimum values of a function occur alternately.

(iv) In Calculus, we are concerned with a relative maximum or a relative minimum value of a function and not with an absolute maximum or absolute minimum values.

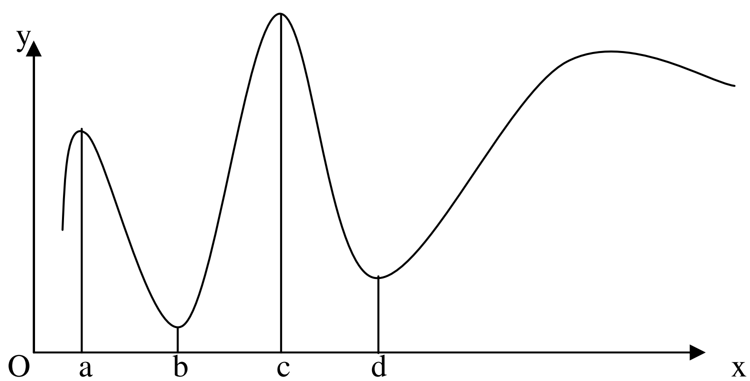
(v) The maximum or minimum point is called a turning point in a curve. The values of the function at these points are called turning values. For the maximum point, the curve ceases to ascend and begins to descend. Such turning value of the function is a maximum. For the minimum point, the curve ceases to descend and begins to ascend and the turning value is a minimum at the turning point.

A function $f(x)$ is said to reach an absolute maximum at $x = a$ if $f(a) > f(x)$ for any other values of x in the domain of $f(x)$.

Absolute and Local Maxima and Minima:

A function $f(x)$ is said to reach an absolute maximum at $x = a$ if $f(a) > f(x)$ for any other values of x in the domain of $f(x)$. On the other hand, a function *f (x)* is said to reach an absolute minimum at $x = a$ if $f(a) < f(x)$ for any other values of x in the domain of $f(x)$.

If a function $f(x)$ is defined on an interval (b,c) which contains $x = a$, $f(x)$ is said to reach a relative (local) maximum at $x = a$ if $f(a) \geq f(x)$ for all other values of x within the interval (b,c). A relative (local) maximum refers to a point where the value of $f(x)$ is greater than values for any other points that are nearby. Again, if a function $f(x)$ is defined on an interval (b,c) that contains $x = a$, $f(x)$ is said to reach a relative (local) minimum at $x = a$ if $f(a) \leq f(x)$ for all other values of x within the interval (b,c). A relative (local) minimum refers to a point where the value of $f(x)$ is lower than values for any other points that are nearby.



From the figure, $f(x)$ reaches an absolute maximum at $x = c$. It reaches an absolute minimum at $x = b$. Again, $f(x)$ has relative maxima at $x = a$ and $x = c$. Similarly, $f(x)$ has relative minima at $x = b$ and $x = d$.

It should be noted that a point on the graph of a function could be both a relative maximum (minimum) and an absolute maximum (minimum).

Thus, local maxima and minima can be determined from the first and second derivatives. The absolute maxima and minima can be found only by comparing the local maxima and local minima with the value of the function at the end points and by selecting the absolute maximum and minimum.

Local maxima and minima can be determined from the first and second derivatives.

Working Rules for Finding of Maximum, Minimum and Point of Inflection of a Function with Single Variable:

Step 1: Differentiate the given function and equate to zero and also find the roots.

(i.e. find $\frac{dy}{dx}$ and put $\frac{dy}{dx} = 0$. Calculate the stationary point.)

Step 2: Again differentiate the given function and put the values of roots in this second derivative function one by one (i.e. compute $\frac{d^2y}{dx^2}$ at these stationary points).

Step 3: If the second derivative is positive for a root then the given function is minimum. On the other hand, if the second derivative is negative for a root then the given function is maximum.

Illustrative Examples:

Example-1:

Let (i) $f(x) = 3x^2 - 14x + 5$

(ii) $f(x) = x^3 - 7x^2 + 6x - 2$

(iii) $f(x) = x^4 - 6x^3 + 4x^2 - 13$

Identify whether the above functions are increasing, decreasing or stationary at $x = 4$.

Solution:

(i) Given that, $f(x) = 3x^2 - 14x + 5$

$$\frac{d(f(x))}{dx} = f'(x) = 6x - 14$$

$$f'(4) = 6 \times 4 - 14 = 10 > 0$$

Thus, the function is increasing.

(ii) Given that, $f(x) = x^3 - 7x^2 + 6x - 2$

$$\frac{d(f(x))}{dx} = f'(x) = 3x^2 - 14x + 6$$

$$f'(4) = 3(4)^2 - 14(4) + 6 = -2 < 0$$

Thus, the function is decreasing.

(iii) Given that, $f(x) = x^4 - 6x^3 + 4x^2 - 13$

$$\frac{d(f(x))}{dx} = f'(x) = 4x^3 - 18x^2 + 8x$$

$$f'(4) = 4(4)^3 - 18(4)^2 + 8(4) = 0$$

Thus, the function is stationary.

Example-2:

Let (i) $f(x) = -2x^3 + 4x^2 + 9x - 15$

(ii) $f(x) = (5x^2 - 8)^2$

Identify whether the above functions are concave or convex at $x = 3$.

Solution:

(i) Given that, $f(x) = -2x^3 + 4x^2 + 9x - 15$

$$\frac{d(f(x))}{dx} = f'(x) = -6x^2 + 8x + 9$$

$$\frac{d^2(f(x))}{dx^2} = f''(x) = -12x + 8$$

$$f''(3) = -12(3) + 8 = -28 < 0$$

Thus, the function is concave.

(ii) Given that, $f(x) = (5x^2 - 8)^2$

$$\frac{d(f(x))}{dx} = f'(x) = 2(5x^2 - 8)(10x) = 100x^3 - 160x$$

$$f''(x) = 300x^2 - 160$$

$$f''(3) = 300(3)^2 - 160 = 2540 > 0$$

Thus, the function is convex.

Example-3:

Find the maximum and minimum values of the function: $x^4 + 2x^3 - 3x^2 - 4x + 4$

Solution:

Let $y = x^4 + 2x^3 - 3x^2 - 4x + 4$.

$$\frac{dy}{dx} = 4x^3 + 6x^2 - 6x - 4$$

Now, if $\frac{dy}{dx} = 0$

then, $4x^3 + 6x^2 - 6x - 4 = 0$

or, $2(x + 2)(2x + 1)(x - 2) = 0$

or, $x = -2, -1/2, 1$.

To find the maximum and minimum values we have to test these values in the second derivative of the function, which is $\frac{d^2y}{dx^2} = 12x^2 + 12x - 6$.

When $x = -2$, $\frac{d^2y}{dx^2} = 12(-2)^2 + 12(-2) - 6 = 66$, which is positive

Hence the given function attains minimum at $x = -2$

And the minimum value is $= x^4 + 2x^3 - 3x^2 - 4x + 4$

$f(2) = (-2)^4 + 2(-2)^3 - 3(-2)^2 - 4(-2) + 4 = 0$

When $x = -1/2$, $\frac{d^2y}{dx^2} = 12(-\frac{1}{2})^2 + 12(-\frac{1}{2}) - 6 = -9$, which is negative

Hence the given function attains maximum at $x = -1/2$

And the maximum value is $= x^4 + 2x^3 - 3x^2 - 4x + 4$

$f(-1/2) = (-\frac{1}{2})^4 + 2(-\frac{1}{2})^3 - 3(-\frac{1}{2})^2 - 4(-\frac{1}{2}) + 4 = \frac{81}{61}$

When $x = 1$, $\frac{d^2y}{dx^2} = 12(1)^2 + 12(1) - 6 = 18$, which is positive

Hence the given function attains minimum at x =1.

And the minimum value is $= x^4 + 2x^3 - 3x^2 - 4x + 4$

$f(1) = (1)^4 + 2(1)^3 - 3(1)^2 - 4(1) + 4 = 0$

Example-4:

Show that the curve $y = x^2(3 - x)$ has a point of inflection at the point (1, 2).

Solution:

We are given that, $y = x^2(3 - x) = 3x^2 - x^3$.

$$\frac{dy}{dx} = 6x - 3x^2$$

$$\frac{d^2y}{dx^2} = 6 - 6x$$

$$\frac{d^3y}{dx^3} = -6$$

For point of inflection, we must have, $\frac{d^2y}{dx^2} = 0$ *and* $\frac{dy^3}{dx^3} \neq 0$

Now, if $\frac{d^2y}{dx^2} = 0$

then, $6 - 6x = 0$

or, $x = 1$

And when $x = 1$, $\frac{dy^3}{dx^3} \neq 0$

When $x = 1$, then $y = 2$.

Hence (1, 2) is the point of inflection.

Example-5:

A sitar manufacturer notices that he can sells x sitars per week at p Taka each where $5x = 375 - 3p$. The cost of production is

$\left(500+13x+\frac{1}{5}x^2\right)$ Taka. Show that the maximum profit is obtained when the production is 30 sitars per week.

Solution:

Given that, $5x = 375 - 3p$

Thus, price $p = \frac{375-5x}{3}$

Revenue = price × quantity

$= \frac{375-5x}{3} \times x$

$= \frac{375x - 5x^2}{3}$

Cost = $\left(500+13x+\frac{1}{5}x^2\right)$

Profit = Revenue – Cost

$P = \frac{375x-5x^2}{3}$ - $\left(500+13x+\frac{1}{5}x^2\right)$

$P = \frac{375x}{3} - \frac{5x^2}{3} - 500 - 13x - \frac{1x^2}{5}$

Differentiate it with respect to *x*,

$\frac{dP}{dx} = \frac{375}{3} - \frac{10x}{3} - 13 - \frac{2x}{5}$

For maxima and minima, $\frac{dP}{dx} = \frac{375}{3} - \frac{10x}{3} - 13 - \frac{2x}{5} = 0$

$x = 30$.

Again,

$\frac{d^2P}{dx^2} = -\frac{10}{3} - \frac{2}{5} = -\frac{56}{15} = -ve$

Thus, the profit function is the maximum at $x = 30$.

Example-6:

A manufacturer sells x units of a product at a dollar price of

$p = p(x) = 6565 - 10x - 0.1x^2$ per unit. The cost of manufacturing the product is

$C(x) = 0.05x^3 - 5x^2 + 20x + 250000, \quad 0 \le x \le 150$. How many units should be produced and sold to maximize the resulting profit?

Solution:

Total revenue **=** (Price per unit) × (Number of units sold)

$$R(x) = (6565 - 10x - 0.1x^2) \times x = 6565x - 10x^2 - 0.1x^3$$

Profit, $P(x)$ = Revenue – Cost

$$= (6565x - 10x^2 - 0.1x^3) - (0.05x^3 - 5x^2 + 20x + 250000)$$

$$= -0.15x^3 - 5x^2 + 6545x - 250000$$

To determine the quantity x that maximizes the profit function, we first find,

$$\frac{dP}{dx} = -0.45x^2 - 10x + 6545$$

Setting this first derivative equals to zero, we solve for critical values of x and $x = 110$ (Other root is outside of the domain).

Now $\frac{d^2P}{dx^2} = -0.9x - 10$

$= -(0.9)(110) - 10 = -109 < 0$; indicating that this critical value does indeed represent a maximum.

P(110) = - 0.15(110)^3 –5(110)^2 + 6545(110) – 250000 = $209800.

Example-7:

A Company has examined its cost structure and revenue structure and has determined that C, the total cost, R, the total revenue and x, the number of units produced are related as: $C = 100 + 0.015x^2$ and $R = 3x$.

Find the production level x that will maximize the profits of the company. Find that profit. Find also the profit when $x = 120$.

Solution:

Profit, $P(x)$ = Revenue – Cost = $R - C$

$$P(x) = 3x - (100 + 0.015x^2) = 3x - 100 - 0.015x^2$$

Hence, $\frac{dP}{dx} = 3 - 0.030x$

For maximum or minimum, $\frac{dP}{dx} = 0$

$$3 - 0.030x = 0$$

$$x = 100 \text{ units}$$

Also $\frac{d^2P}{dx^2} = -0.030 < 0$

Thus, profit will be maximum when $x = 100$ and the maximum profit

$$P = 3\times100 - 100 - 0.015\times(100)^2 = 50$$

When $x = 120$, Profit, $P = 3\times120 - 100 - 0.015\times(120)^2 = 44$.

Example-8:

The total cost function of a firm is given by $C = \frac{1}{3}x^3 - 5x^2 + 28x + 10$ where C is the total cost and x is the output of the product. A tax at the rate of $2 per unit of product is imposed and the producer adds it to his cost. If the market demand function is given by $p = 2530 - 5x$, where p is the price per unit of output, find the profit maximizing output and price.

Solution:

Profit, $P(x)$ = Total Revenue – Total Cost = TR – TC

$$P(x) = \{(2530 - 5x)x\} - \{(\frac{1}{3}x^3 - 5x^2 + 28x + 10) + 2x\}$$

$$= (2530x - 5x^2) - (\frac{1}{3}x^3 - 5x^2 + 30x + 10)$$

Hence, $\frac{dP}{dx} = (2530 - 10x) - (x^2 - 10x + 30)$.

For maximum or minimum, $\frac{dP}{dx} = 0$

$$(2530 - 10x) - (x^2 - 10x + 30) = 0$$

$x^2 = 2500$ units

$x = 50$ (since production level cannot be negative, so ignoring –ve sign)

Also $\frac{d^2P}{dx^2} = -2x < 0$

Thus, profit will be maximum when $x = 50$ units and the maximum profit

$$P = 2530 - 5\times50 = \$2280$$

Questions for Review:

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. Identify whether the following function is increasing, decreasing or stationary at $x = 5$. Find also identify whether the function is concave or convex at $x = 5$.

   $$f(x) = 2x^3 - 30x^2 + 126x + 59.$$

2. A firm has determined that its weekly profit function is given by

   $$P(x) = 95x - 0.05x^2 - 5000; \quad 0 \le x \le 1000$$

   where $P(x)$ is the profit in dollars and x is the number of units of the product sold.
   For what value of x does profit reach a maximum? What is this maximum profit?

3. A study has shown that the cost of producing Orange Juice of a manufacturing concern is given by $C = 30 + 1.5x + 0.0008x^2$. What is the marginal cost at $x = 1000$?

   If the Juice sells for Tk.5 each for what values of x does marginal cost equal marginal revenue? [Hint: Marginal cost is the value of dC/dx at $x = 1000$.]

Lesson-2: Optimization of Multivariate Functions

After completing this lesson, you should be able to:

- Express the concept of optimization with multivariate function;
- Determine the maximum and minimum point of a multivariate function.

Introduction:

Many economic activities involve functions of more than one independent variable. Let $Z = f(x, y)$ and $P = f(x, y, z)$ are defined as functions of two and three independent variables respectively. In order to measure the effect of a change in a single independent variable on the dependent variable in a multivariate function, the partial derivative is needed. Partial derivative with respect to one of the independent variables follows the same rules as ordinary differentiation while the other independent variables are treated as constant. This lesson extends the ideas of relative maximum and minimum for functions of one variable to multivariate functions.

Partial derivative with respect to one of the independent variables follows the same rules as ordinary differentiation while the other independent variables are treated as constant.

Determination of the Maximum and Minimum Values of a Function with Two Independent Variables:

Let $\phi(x, y)$ be a function of two independent variables *x* and *y*. We are to investigate at (a,b) whether $\phi(x, y)$ is maximum or minimum.

For the existence of a maximum or a minimum at (a,b)

$$\frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial y} = 0$$

Again let, $A = \dfrac{\partial^2\phi}{\partial x^2}$, $H = \dfrac{\partial^2\phi}{\partial x\partial y}$, $B = \dfrac{\partial^2\phi}{\partial y^2}$.

Case 1: If $AB - H^2$ is positive. $\phi(a,b)$ is maximum or minimum if *A* and *B* are both negative or are both positive respectively.

Case 2: If $AB - H^2$ is negative, $\phi(a,b)$ is neither a maximum nor a minimum.

Case 3: If $AB < H^2$ when *A* and *B* have the same signs, the function is at an inflection point; when *A* and *B* have different signs, the function is at a saddle point.

Case 4: If $AB = H^2$, the test is inconclusive.

Illustrative Examples:

Example-1:

Determine the values of *x* and *y* for which, $Z = 4x^2 + 2y^2 + 10x - 6y - 4xy$ is an optimum; specify whether the optimum is a maximum or minimum. Calculate the value of the function at the optimum.

Solution:

Given that, $Z = 4x^2 + 2y^2 + 10x - 6y - 4xy$

$$\frac{\partial z}{\partial x} = 8x - 4y + 10 = 0 \quad \text{.....................(i)}$$

$$\frac{\partial z}{\partial y} = -4x + 4y - 6 = 0 \quad \text{...................(ii)}$$

Solving equations (i) and (ii) simultaneously for x and y gives $x = -1$ and $y = \frac{1}{2}$

Again, $A = \frac{\partial^2 z}{\partial x^2} = 8$, $B = \frac{\partial^2 z}{\partial y^2} = 4$, $H = \frac{\partial^2 z}{\partial x \partial y} = -4$

$AB - H^2 = (8)(4) - (-4)^2 = 16 > 0.$

Since the second derivatives are positive and the product of the second derivatives is greater than the square of the cross partial derivative, the function reaches a relative minimum at $x = -1$ and $y = \frac{1}{2}$.

The value of the function $Z = 4x^2 + 2y^2 + 10x - 6y - 4xy$ at the optimum is,

$$Z(-1, \frac{1}{2}) = 4(-1)^2 + 2(\frac{1}{2})^2 + 10(-1) - 6(\frac{1}{2}) - 4(-1)(\frac{1}{2}) = -6\frac{1}{2}$$

Example-2:

Determine the values of x and y for which, $Z = -4x^2 + 4xy - 2y^2 + 16x - 12y$ is an optimum; specify whether the optimum is a maximum or minimum. Calculate the value of the function at the optimum.

Solution:

Given that, $Z = -4x^2 + 4xy - 2y^2 + 16x - 12y$

$$\frac{\partial z}{\partial x} = -8x + 4y + 16 = 0 \quad \text{.......................(i)}$$

$$\frac{\partial z}{\partial y} = 4x - 4y - 12 = 0 \quad \text{........................(ii)}$$

Solving equations (i) and (ii) simultaneously for x and y gives $x = 1$ and $y = -2$

Again, $A = \frac{\partial^2 z}{\partial x^2} = -8$, $B = \frac{\partial^2 z}{\partial y^2} = -4$, $H = \frac{\partial^2 z}{\partial x \partial y} = 4$

$AB - H^2 = (-8)(-4) - (4)^2 = 16 > 0.$

Since the second derivatives are negative and the product of the second derivatives is greater than the square of the cross partial derivative, the function reaches a relative maximum at $x = 1$ and $y = -2$.

The value of the function $Z = -4x^2 + 4xy - 2y^2 + 16x - 12y$ at the optimum is:

$$Z(1, -2) = -4(1)^2 + 4(1)(-2) - 2(-2)^2 + 16(1) - 12(-2) = 20$$

Example-3:

The total production cost of a product is given by $f(x, y) = x^2 + y^2 - 5x - 9y - xy + 90$.

where $f(x, y)$ = Total cost in thousand dollar.

x = Number of labor hours used (in hundred).

y = Number of pounds for raw materials used (in hundred).

It is required to determine how many labor hours and how many pounds of raw materials should be used in order to minimize the total cost.

Solution:

Given that, $f(x, y) = x^2 + y^2 - 5x - 9y - xy + 90$

The partial derivatives are

$$\frac{\partial f}{\partial x} = 2x - 5 - y = 0 \quad \text{..................(i)}$$

$$\frac{\partial f}{\partial y} = 2y - 9 - x = 0 \quad \text{..................(ii)}$$

Solving equations (i) and (ii) simultaneously for x and y gives $x = \frac{19}{3}$ and $y = \frac{23}{3}$

Again, the second order partial derivatives are, $A = \frac{\partial^2 f}{\partial x^2} = 2$, $B = \frac{\partial^2 f}{\partial y^2} = 2$, $H = \frac{\partial^2 f}{\partial x \partial y} = -1$

$$AB - H^2 = (2)(2) - (-1)^2 = 3 > 0.$$

Since the second derivatives are positive and the product of the second derivatives is greater than the square of the cross partial derivative, the function reaches a relative minimum at $x = \frac{19}{3}$ and $y = \frac{23}{3}$.

The total cost will be minimum when $^{19}/_3$ hundred (633) labor hours and $^{23}/_3$ hundred (767) pounds of raw materials are used. The total cost with this production strategy is:

$$f\left(\frac{19}{3}, \frac{23}{3}\right) = \left(\frac{19}{3}\right)^2 + \left(\frac{23}{3}\right)^2 - 5\left(\frac{19}{3}\right) - 9\left(\frac{23}{3}\right) - \left(\frac{19}{3}\right)\left(\frac{23}{3}\right) + 90$$

= 39.667 thousand or 39,667.

Determination of the Maximum and Minimum Values of a Function with Three Independent Variables:

Let $\phi(x, y, z)$ be a function of three independent variables *x*, *y* and *z*. We are to investigate at (a,b,c) whether $\phi(x, y, z)$ is maximum or minimum.

For the existence of a maximum or a minimum at (a,b,c)

$$\frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial y} = \frac{\partial\phi}{\partial z} = 0$$

Again let $A = \frac{\partial^2\phi}{\partial x^2}$, $B = \frac{\partial^2\phi}{\partial y^2}$, $C = \frac{\partial^2\phi}{\partial z^2}$, $F = \frac{\partial^2\phi}{\partial y\partial z}$, $G = \frac{\partial^2\phi}{\partial z\partial x}$,

$$H = \frac{\partial^2\phi}{\partial x\partial y}$$

Case 1: $\phi(a,b,c)$ is minimum if *A*, $\begin{vmatrix} A & H \\ H & B \end{vmatrix}$, $\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}$ are all positive.

Case 2: $\phi(a,b,c)$ is maximum if *A*, $\begin{vmatrix} A & H \\ H & B \end{vmatrix}$, $\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}$ are alternately negative and positive.

Case 3: If the above conditions are not satisfied, then $\phi(a,b,c)$ is neither a maximum nor a minimum.

Illustrative Examples:

Example-4:

Show that the function $\phi(x, y, z) = x^2 + y^2 + z^2 + x - 2z - xy$ has a minimum value at $(-2/3, -1/3, 1)$

Solution:

We are given that, $\phi(x, y, z) = x^2 + y^2 + z^2 + x - 2z - xy$.

$$\frac{\partial\phi}{\partial x} = 2x + 1 - y\,.$$

$$\frac{\partial\phi}{\partial y} = 2y - x$$

$$\frac{\partial\phi}{\partial z} = 2z - 2\,.$$

$\phi(x, y, z)$ will be maximum or minimum if $\frac{\partial\phi}{\partial x} = 0$, $\frac{\partial\phi}{\partial y} = 0$, $\frac{\partial\phi}{\partial z} = 0$.

That is $$\frac{\partial\phi}{\partial x} = 2x + 1 - y = 0 \quad \text{.............(i)}$$

$$\frac{\partial\phi}{\partial y} = 2y - x = 0 \quad \text{...............(ii)}$$

$$\frac{\partial\phi}{\partial z} = 2z - 2 = 0 \quad \text{.................(iii)}$$

Solving equations (i), (ii) and (iii) simultaneously, we get $x = -{}^2/_3$, $y = -{}^1/_3$ and $z = 1$.

Again, $A = \frac{\partial^2\phi}{\partial x^2} = 2$, $B = \frac{\partial^2\phi}{\partial y^2} = 2$, $C = \frac{\partial^2\phi}{\partial z^2} = 2$, $F = \frac{\partial^2\phi}{\partial y\partial z} = 0$,

$$G = \frac{\partial^2\phi}{\partial z\partial x} = 0, \; H = \frac{\partial^2\phi}{\partial x\partial y} = -1.$$

Since, A = 2, $\begin{vmatrix} A & H \\ H & B \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3$ and

$$\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 6$$ are all positive.

Hence, the given function $\phi(x, y, z)$ is minimum at $\left(-\frac{2}{3}, -\frac{1}{3}, 1\right)$.

Example-5:

Optimize the following function:

$$y = -5x_1^2 + 10x_1 + x_1x_3 - 2x_2^2 + 4x_2 + 2x_2x_3 - 4x_3^2$$

Solution:

We are given that,

$$\phi(x_1, x_2, x_3) = y = -5x_1^2 + 10x_1 + x_1x_3 - 2x_2^2 + 4x_2 + 2x_2x_3 - 4x_3^2$$

$$\frac{\partial\phi}{\partial x_1} = -10x_1 + 10 + x_3 = 0.$$

$$\frac{\partial\phi}{\partial x_2} = -4x_2 + 2x_3 + 4 = 0$$

$$\frac{\partial\phi}{\partial x_3} = x_1 + 2x_2 - 8x_3 = 0.$$

Solving the above three equations by using Cramer's Rule we get $x_1 = 1.04$, $x_2 = 1.22$ and $x_3 = 0.43$.

Again, $A = \frac{\partial^2 \phi}{\partial x_1^2} = -10$, $B = \frac{\partial^2 \phi}{\partial x_2^2} = -4$, $C = \frac{\partial^2 \phi}{\partial x_3^2} = -8$, $F = \frac{\partial^2 \phi}{\partial x_2 \partial x_3} = 2$,

$$G = \frac{\partial^2 \phi}{\partial x_3 \partial x_1} = 1,\ H = \frac{\partial^2 \phi}{\partial x_1 \partial x_2} = 0.$$

Since, $A = -10$, $\begin{vmatrix} A & H \\ H & B \end{vmatrix} = \begin{vmatrix} -10 & 0 \\ 0 & -4 \end{vmatrix} = 40$ and

$$\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = \begin{vmatrix} -10 & 0 & 1 \\ 0 & -4 & 2 \\ 1 & 2 & -8 \end{vmatrix} = -276$$

Since the principal minors alternate correctly in sign, hence the given function is maximized at $x_1 = 1.04$, $x_2 = 1.22$ and $x_3 = 0.43$.

Questions for Review:

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. Find the maxima and minima for the following functions:

   (a) $f(x, y) = 2x^2 - 4x + y^2 - 4y + 4$.

   (b) $f(x, y) = 2x^2 - 4x + 8y^2 + 80y + 50xy + 100$.

2. A company produces two products, x units of type A and y units of type B per month. If the revenue and cost equations for the month are given by $R(x, y) = 11x + 14y$ and $C(x, y) = x^2 - xy + 2y^2 + 3x + 4y + 10$, find the state (x, y) that yields maximum profit.

3. The cost of construction C of a project depends upon number of skilled workers *(x)* and unskilled workers *(y)*. It is given that cost $C(x, y) = 40{,}000 + 9x^3 - 72xy + 9y^2$. Determine the number of skilled and unskilled workers results in minimum cost. Find also the minimum cost.

4. The yearly profits of a small service organization Econo Ltd. are dependent upon the number of workers *(x)* and the number of units of advertising *(y)*, and the profit function is, $P(x, y) = 412x + 806y - x^2 - 5y^2 - xy$. Determine the number of workers and the number of units in advertising that results in maximum profits. Determine also the maximum profits.

Lesson-3: Constrained Optimization with Lagrangian Multipliers

After completing this lesson, you should be able to:

- Express the concept of constrained optimization;
- Determine the maximum point of a function with some constraints;
- Determine the minimum point of a function with some constraints.

Introduction

A decision maker must consider different types of physical and legal restrictions.

Decision makers do not normally have unlimited resources for their use. A decision maker must consider different types of physical and legal restrictions. Solutions to economic problems often have to be found under constraints, e.g., maximizing utility subject to a budget constraint or minimizing costs subject to some such minimal requirement of output as a production quota etc. Classical differential calculus is used to optimize (maximize or minimize) a function subject to constraint. In this respect, use of the Lagrangian function greatly facilitates this task.

Lagrangian multipliers provide a method of determining the optimum value of a differentiable nonlinear function subject to linear or nonlinear constraints.

The Lagrangian multiplier (λ) approximates the marginal impact on the objective function caused by a small change in the constant of the constraint. Lagrangian multipliers provide a method of determining the optimum value of a differentiable nonlinear function subject to linear or nonlinear constraints. The method of Lagrangian multipliers is useful in allocating scarce resources between alternative uses. In this lesson, we shall now consider procedures used for determining relative maxima and minima for a multivariate function on which certain constraints are imposed.

Working Rule for Constrained Optimization with Lagrange Multiplier:

Step 1: Given a function $f(x, y)$ subject to a constraint $g(x, y) = K$ (a constant), a new function F can be formed by setting the constraint equal to zero.

Step 2: Multiplying it by λ (the Lagrange multiplier) and adding the product to the original function:

$$F(x, y, \lambda) = f(x, y) + \lambda[K - g(x, y)]$$

Step 3: Critical values x, y, λ at which the function is optimized, are found by taking the partial derivatives of F with respect to all three independent variables, setting them equal to zero and solving simultaneously:

$$F_x(x, y, \lambda) = 0,\ F_y(x, y, \lambda) = 0,\ F_\lambda(x, y, \lambda) = 0,$$

Illustrative Examples:

Example -1:

Determine the critical points and the constrained optimum for $Z = 4x^2 - 2xy + 6y^2$; subject to $x + y = 72$.

Solution:

The Lagrangian function is

$$F(x, y, \lambda) = f(x, y) + \lambda[K - g(x, y)]$$
$$= 4x^2 - 2xy + 6y^2 + \lambda(72 - x - y)$$

The partial derivatives are

$$\frac{\partial F}{\partial x} = 8x - 2y - \lambda = 0.$$
$$\frac{\partial F}{\partial y} = -2x + 12y - \lambda = 0.$$
$$\frac{\partial F}{\partial \lambda} = 72 - x - y = 0.$$

The above three equations are solved simultaneously and we get $x = 42,\ y = 30$ and $\lambda = 276$.

Thus Z = [4 $(42)^2$ – 2 (42) (30) + 6 $(30)^2$ + 276 (72 – 42 – 30)] = 9936.

With the Lagrangian multiplier λ = 276 means that one unit increase in the constant of the constraint will lead to an increase of approximately 276 in the value of the objective function and Z ≈ 10212.

Example-2:

Determine the critical points and the constrained optima for $Z = x^2 + 3xy + y^2$; subject to $x + y = 100$.

Solution:

The Lagrangian function is

$$F(x, y, \lambda) = f(x, y) + \lambda[K - g(x, y)]$$
$$= x^2 + 3xy + y^2 + \lambda(100 - x - y)$$

The partial derivatives are

$$\frac{\partial F}{\partial x} = 2x + 3y - \lambda = 0.$$
$$\frac{\partial F}{\partial y} = 3x + 2y - \lambda = 0.$$
$$\frac{\partial F}{\partial \lambda} = 100 - x - y = 0.$$

The above three equations are solved simultaneously and we get $x = 50,\ y = 50$ and $\lambda = -250$. Thus, the constrained optimum,

Z = [$(50)^2$ + 3 (50) (50) + $(50)^2$ - 250 (100 – 50 – 50)] = 12500.

To determine if the function reaches a maximum or a minimum, we evaluate the function at points adjacent to x = 50 and y = 50. The function is a constrained maximum since adding Δx and Δy to the function in both directions gives a functional value less than the constrained optimum.

That is F (49, 51, –250) = 12,499

F (51, 49, –250) = 12,499.

Example-3:

What output mix should a profit maximizing firm produce when its total profit function is $\pi = 80x - 2x^2 - xy - 3y^2 + 100y$ and its maximum output capacity is $x + y = 12$? Estimate also the effect on profits if output capacity is expanded by one unit.

Solution:

The Lagrangian function is

$$F(x, y, \lambda) = f(x, y) + \lambda[K - g(x, y)]$$

$$= 80x - 2x^2 - xy - 3y^2 + 100y + \lambda(12 - x - y)$$

The partial derivatives are

$$\frac{\partial F}{\partial x} = 80 - 4x - y - \lambda = 0.$$

$$\frac{\partial F}{\partial y} = -x - 6y + 100 - \lambda = 0.$$

$$\frac{\partial F}{\partial \lambda} = 12 - x - y = 0.$$

The above three equations are solved simultaneously and we get $x = 5,\ y = 7$ and $\lambda = 53$.

Thus profit, $\pi = [80(5) - 2(5)^2 - (5)(7) - 3(7)^2 + 100(7) = 868$.

Hence, λ=53 means that an increase in output capacity should lead to increased profits of approximately 53.

Inequality Constraints:

The method of Lagrangian multipliers can be modified to incorporate constraints that take the form of inequalities rather than equalities.

The method of Lagrangian multipliers can be modified to incorporate constraints that take the form of inequalities rather than equalities. The problem now becomes that of determining the extreme points of the multivariate function $z = f(x, y)$ subject to the inequality $g(x, y) \le 0$ or $g(x, y) \ge 0$. We shall now a relatively simple extension of the Lagrangian technique that provides a solution to the problem of optimizing an objective function subject to single constraining inequality. The general problem of optimizing an objective function subject to n inequalities is not considered in this lesson.

In the problem of optimizing an objective function subject to a constraining inequality, two cases must be considered. Firstly, the constraining equality may act as an upper or lower bound on the function. Secondly, the constraining inequality does not act as an upper or lower bound on the function. For either of the above two cases, the method of optimizing a function subject to a constraining inequality is to assume that the constraining inequality is equality; i.e. assume $g(x, y) = 0$. The inequality is changed to equality and the critical values and constrained optimum are obtained by using the method of Lagrangian multipliers. The sign of the Lagrangian multiplier is used to determine whether the constraint is actually limiting the optimum value of the objective function. The procedure is as follows:

Case 1: When the constraining inequality $g(x, y) \leq 0$:

For maximizing the objective function

(i) If $\lambda > 0$, the restriction is not a limitation; we resolve the problem ignoring the restriction, to obtain the optimum.

(ii) If $\lambda \leq 0$, the restriction acts as an upper bound and the result can be obtained by assuming that $g(x, y) = 0$ is the constrained optimum.

For minimizing the objective function

(i) If $\lambda > 0$, the restriction is a limitation and the constrained optimum is obtained by assuming that $g(x, y) = 0$.

(ii) If $\lambda \leq 0$, the restriction is not a limitation; we resolve the problem ignoring the restriction, to obtain the optimum.

Case 2: When the constraining inequality $g(x, y) \geq 0$:

For maximizing the objective function

(i) If $\lambda > 0$, the restriction is a limitation and the constrained optimum is obtained by assuming that $g(x, y) = 0$.

(ii) If $\lambda \leq 0$, the restriction is not a limitation; we resolve the problem ignoring the restriction, to obtain the optimum.

For minimizing the objective function

(i) If $\lambda > 0$, the restriction is not a limitation; we resolve the problem ignoring the restriction, to obtain the optimum.

(ii) If $\lambda \leq 0$, the restriction is a limitation and the constrained optimum is obtained by assuming that $g(x, y) = 0$.

The following examples illustrate this technique.

Illustrative Example:

Example-4:

Determine the maximum of the function $z = x^2 + 3xy + y^2$
subject to $x + y \leq 100$.

Solution:

Given that the objective function is $z = x^2 + 3xy + y^2$ ....(i)

the constraint equation is $x + y \leq 100$. ....(ii)

To determine the solution we treat the inequality as equality and form the Lagrangian expression

$$F(x, y, \lambda) = x^2 + 3xy + y^2 + \lambda[x + y - 100]$$

The critical values are $x^* = 50$, $y^* = 50$ and $\lambda^* = -250$.

Since the Lagrangian multiplier is negative, an increase in the constant of the constraining function would increase the value of the objective function. Therefore, the value of the function is limited by the constraint.

Example-5:

Determine the maximum profit of the following profit function $z = 600 - 4x^2 + 20y + 2xy - 6y^2 + 12x$ subject to $x + y \geq 5$. Determine the values of *x* and *y* that maximizes the profits. Assume that *x* and *y* in million dollar and profits in thousand dollar respectively.

Solution:

Given that the objective function is

$$z = 600 - 4x^2 + 20y + 2xy - 6y^2 + 12x \text{ ....(i)}$$

and the constraint equation is $x + y \geq 5$. ....(ii)

The Lagrangian expression is

$$F(x, y, \lambda) = 600 - 4x^2 + 20y + 2xy - 6y^2 + 12x + \lambda[x + y - 5]$$

The partial derivatives of this expression are equal to zero.

$$\frac{\partial F}{\partial x} = -8x + 2y + 12 + \lambda = 0 \text{ ....(iii)}$$

$$\frac{\partial F}{\partial y} = 2x - 12y + 20 + \lambda = 0 \text{ ....(iv)}$$

$$\frac{\partial F}{\partial \lambda} = x + y - 5 = 0 \text{ ....(v)}$$

Solving equations (iii), (iv) and (v) simultaneously we get,

$$x^* = 2.584, \quad y^* = 2.416 \quad \text{and} \quad \lambda^* = 3.834.$$

.

Thus, the maximum profit will be z = [600 – 4(2.584)^2 + 20(2.416) + 2 (2.584)(2.416) – 6(2.416)^2 + 12(2.584)] = 630.083 thousand or $630,083. Profits, subject to the constraint, are maximized when *x* = $2.584 million ($2584000), *y* = $2.416 million ($2416000) are obtained. Since λ is positive, a decrease in the constraint constant will increase the value of the objective function. Therefore, the constraint acts as a lower bound on the values of the variables.

Questions for Review:

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. Determine the critical points and the constrained optimum for $f(x, y) = 12xy - 3y^2 - x^2$ subject to $x + y = 16$.
2. Determine the critical points and the constrained optimum for $f(x, y) = 3x^2 + 4y^2 - xy$ subject to $2x + y = 21$.
3. Determine the maximum of the function $z = 10xy - 5x^2 - 7y^2 + 40x$ subject to $x + y \le 12$.
4. Determine the minimum of the function $z = 4x^2 + 5y^2 - 6y$ subject to $x + 2y \ge 20$.
5. A firm's total costs can be presented as $C = 3x^2 + 5xy + 6y^2$, find the firm's minimum cost to meet a production quota of $5x + 7y = 732$.

Integral Calculus 10

This unit is designed to introduce the learners to the basic concepts associated with Integral Calculus. Integral calculus can be classified and discussed into two threads. One is ***Indefinite Integral*** and the other one is ***Definite Integral***. The learners will learn about indefinite integral, methods of integration, definite integral and application of integral calculus in business and economics.

Blank Page

Lesson-1: Indefinite Integral

After completing this lesson, you should be able to:

- Describe the concept of integration;
- Determine the indefinite integral of a given function.

Introduction

The process of differentiation is used for finding derivatives and differentials of functions. On the other hand, the process of integration is used (i) for finding the limit of the sum of an infinite number of infinitesimals that are in the differential form $f'(x)dx$ (ii) for finding functions whose derivatives or differentials are given, i.e., for finding anti-derivatives. Thus, reversing the process of differentiation and finding the original function from the derivative is called integration or anti-differentiation.

Reversing the process of differentiation and finding the original function from the derivative is called integration.

The integral calculus is used to find the areas, probabilities and to solve equations involving derivatives. Integration is also used to determine a function whose rate of change is known.

In integration whether the object be summation or anti-differentiation, the sign $\int$, an elongated *S*, the first letter of the word 'sum' is most generally used to indicate the process of the summation or integration. Therefore, $\int f(x)dx$ is read the integral of *f (x)* with respect to *x*.

$\int f(x)dx$ is read the integral of f (x) with respect to x.

Again, integration is defined as the inverse process of differentiation.

Thus if $\frac{d}{dx}g(x) = f(x)$

then $\int f(x)dx = g(x) + c$

where *c* is called the constant of integration. Of course *c* could have any value and thus integral of a function is not unique! But we could say one thing here that any two integrals of the same function differ by a constant. Since *c* could also have the value zero, $g(x)$ is one of the values of $\int f(x)dx$. As *c* is unknown and indefinite, hence it is also referred to as Indefinite Integral.

Some Properties of Integration

The following two rules are useful in reducing differentiable expressions to standard forms.

(i) The integral of any algebraic sum of differential expression equals the algebraic sum of the integrals of these expressions taken separately.

i.e. $\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$

(ii) A constant multiplicative term may be written either before or after the integral sign.

i.e. $\int cf(x)dx = c\int f(x)dx$; where c is a constant.

Some Standard Results of integration

A list of some standard results by using the derivative of some well-known functions is given below:

(i) $\int dx = x + c$ $\therefore \frac{d}{dx}(x) = 1$

(ii) $\int x^n dx = \frac{x^{n+1}}{n+1} + c$ $\therefore \frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n, \quad n \neq -1$

(iii) $\int \frac{1}{x} dx = \log x + c$ $\therefore \frac{d}{dx}(\log x) = \frac{1}{x}$

(iv) $\int e^x dx = e^x + c$ $\therefore \frac{d}{dx}(e^x) = e^x$

(v) $\int a^x dx = \frac{a^x}{\log a} + c$ $\therefore \frac{d}{dx}(a^x) = a^x \log a$

(vi) $\int \sin x \, dx = -\cos x + c$ $\therefore \frac{d}{dx}(-\cos x) = \sin x$

(vii) $\int \cos x \, dx = \sin x + c$ $\therefore \frac{d}{dx}(\sin x) = \cos x$

(viii) $\int \sec^2 x \, dx = \tan x + c$ $\therefore \frac{d}{dx}(\tan x) = \sec^2 x$

(ix) $\int \operatorname{cosec}^2 x \, dx = -\cot x + c$ $\therefore \frac{d}{dx}(-\cot x) = \operatorname{cosec}^2 x$

(x) $\int \sec x \tan x \, dx = \sec x + c$ $\therefore \frac{d}{dx}(\sec x) = \sec x \tan x$

(xi) $\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + c$

$\therefore \frac{d}{dx}(-\operatorname{cosec} x) = \operatorname{cosec} x \cot x$

(xii) $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + c$ $\therefore \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$

(xii) $\int \frac{1}{1+x^2} dx = \tan^{-1} x + c$ $\therefore \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

(xiii) $\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x + c$ $\therefore \frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{1+x^2}}$

(xiv) $\int \tan x \, dx = \log \sec x + c$ $\therefore \frac{d}{dx}(\log \sin x) = \cot x$

(xv) $\int \cot x \, dx = \log \sin x + c$ $\therefore \frac{d}{dx}(\log \sec x) = \tan x$

Illustrative Examples:

Example -1:

Evaluate $\int x^2\,dx$

Solution:

$$\int x^2\,dx = \frac{x^3}{3} + c$$

Example-2:

Evaluate $\int 5x\,dx$

Solution:

$$\int 5x\,dx = \frac{5}{2}x^2 + c$$

Example-3:

Evaluate $\int -2\,dx$

Solution:

$$\int -2\,dx = -2x + c$$

Example-4:

Evaluate $\int (2x+3)\,dx$

Solution:

$$\int (2x+3)\,dx = \int 2x dx + \int 3dx$$

$$= x^2 + 3x + c$$

Example-5:

Evaluate $\int (4x^2 - 7x + 6)\,dx$

Solution:

$$\int (4x^2 - 7x + 6)\,dx = \int 4x^2 dx - \int 7x dx + \int 6dx$$

$$= \frac{4}{3}x^3 - \frac{7}{2}x^2 + 6x + c$$

Example-6:

Evaluate $\int \left(\frac{\mathrm{a}}{x^3} + \frac{b}{x^2} + 6\right)dx$

Solution:

$$\int \left(\frac{\mathrm{a}}{x^3} + \frac{b}{x^2} + 6\right)dx = \int \frac{a}{x^3}dx + \int \frac{b}{x^2}dx + \int 6dx$$

$$= -\frac{a}{2x^2} - \frac{b}{x} + 6x + c$$

Example-7:

Evaluate $\int \frac{dx}{\sqrt{x-1} + \sqrt{x+3}}$

Solution:

$$\int \frac{dx}{\sqrt{x-1} + \sqrt{x+3}} = \int \frac{(\sqrt{x-1} - \sqrt{x+3})\,dx}{(\sqrt{x-1} + \sqrt{x+3})(\sqrt{x-1} - \sqrt{x+3})}$$

$$= \int \frac{(\sqrt{x-1} - \sqrt{x+3})\,dx}{x-1-x-3}$$

$$= \frac{-1}{4}\int \sqrt{x-1}\,dx + \frac{1}{4}\int \sqrt{x+3}\,dx$$

$$= \frac{1}{6}(x+3)^{\frac{3}{2}} - \frac{1}{6}(x-1)^{\frac{3}{2}} + c$$

Example-8:

Evaluate $\int (5e^x - x^{-2} + \frac{3}{x})\,dx; \quad x \neq 0$

Solution:

$$\int (5e^x - x^{-2} + \frac{3}{x})\,dx = \int 5e^x dx - \int x^{-2} dx + \int \frac{3}{x} dx$$

$$= 5e^x + \frac{1}{x} + 3\log x + c$$

Questions for Review:

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. Integrate the following functions w. r. t. x

   (i) $\int (5x^3 + \frac{6}{x^3})dx$

   (ii) $\int \frac{1}{\sqrt{x}} dx$

   (iii) $\int (\sqrt{x} - \frac{1}{\sqrt{x}})dx$

   (iv) $\int (x^3 + \frac{1}{x^3})dx$

   (v) $\int \frac{dx}{\sqrt{x+1} + \sqrt{x-1}}$

   (vi) $\int (2e^x + 7^x + \frac{1}{\sqrt{x}})dx$

Lesson-2: Methods of Integration

After studying this lesson, you should be able to:

- Describe the methods of integration;
- Determine the integral of any function.

Introduction

In comparing integral and differential calculus, most of the mathematicians would agree that the integration of functions is a more complicated process than the differentiation of functions. Functions can be differentiated through application of a number of relatively straightforward rules. This is not true in determining the integrals of functions. Integration is much less straightforward and often requires considerable ingenuity.

Certain functions can be integrated quite simply by applying rules of integration.

Certain functions can be integrated quite simply by applying rules of integration. A natural question is that what happens when rules of integration cannot be applied directly. Such functions require more complicated techniques. This lesson discusses four techniques that can be employed when the other rules do not apply and when the structure of the integrand is of an appropriate form. In general, experience is the best guide for suggesting the quickest and simplest method for integrating any given function.

Methods of Integration

The following are the four principal methods of integration:

(i) Integration by substitution;

(ii) Integration by parts;

(iii) Integration by successive reductions;

(iv) Integration by partial fraction.

Integration by Substitution

Substituting a new suitable variable for the given independent variable and integrating with respect to the substituted variable can often facilitate Integration.

Substituting a new suitable variable for the given independent variable and integrating with respect to the substituted variable can often facilitate Integration. Experience is the best guide as to what substitution is likely to transform the given expression into another that is more readily integrable. In fact this is done only for convenience.

The following examples will make the process clear.

Example-1:

Evaluate $\int (ax+b)^5 dx$

Solution:

Let, $ax+b=z$

$$adx = dz$$

$$dx = \frac{1}{a}dz$$

$$\int (ax+b)^5 dx = \int z^5 . \frac{1}{a} dz$$

$$= \frac{1}{a}\int z^5 dz$$

$$= \frac{1}{a}.\frac{z^6}{6} + c$$

$$= \frac{1}{a}.\frac{(ax+b)^6}{6} + c$$

Example-2:

Evaluate $\int x(x^2+4)^5 dx$

Solution:

Let, $x^2 + 4 = z$

$2xdx = dz$

$xdx = \frac{1}{2}dz$

$$\int x(x^2+4)^5 dx = \int z^5.\frac{1}{2}dz$$

$$= \frac{1}{2}\int z^5 dz$$

$$= \frac{1}{2}.\frac{z^6}{6} + c$$

$$= \frac{1}{12}(x^2+4)^6 + c$$

Example-3:

Evaluate $\int x^2 e^{x^3} dx$

Solution:

Let, $x^3 = z$

$3x^2 dx = dz$

$x^2 dx = \frac{1}{3}dz$

$$\int x^2 e^{x^3} dx = \int e^z.\frac{1}{3}dz$$

$$= \frac{1}{3}e^z + c$$

$$= \frac{1}{3}e^{x^3} + c$$

Example-4:

Evaluate $\int x\sqrt{x^2+1}\, dx$.

Solution:

Let, $x^2 + 1 = z$

$2xdx = dz$

$$xdx = \frac{1}{2}dz$$

$$\int x\sqrt{x^2+1}\,dx = \int \sqrt{z}.\ \frac{1}{2}dz$$

$$= \frac{1}{2}\int z^{\frac{1}{2}}dz$$

$$= \frac{1}{2}.\frac{z^{\frac{3}{2}}}{\frac{3}{2}} + \text{c}$$

$$= \frac{1}{3}(x^2+1)^{\frac{3}{2}} + c$$

Integration by Parts

Integration by parts is a special method that can be applied in finding the integrals of a product of two integrable functions.

Integration by parts is a special method that can be applied in finding the integrals of a product of two integrable functions. This method of integration is derived from the rule of differentiation of a product of two functions.

If u and v are two functions of x then,

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$u\frac{dv}{dx} = \frac{d}{dx}(uv) - v\frac{du}{dx}$$

Integrating both sides with respect to x, we get

$$\int u\frac{dv}{dx}dx = \int \frac{d}{dx}(uv)dx - \int v\frac{du}{dx}dx$$

$$\int u\frac{dv}{dx}dx = \int uv - \int v\frac{du}{dx}dx$$

Putting $\text{u} = \text{f(x)}$, $\frac{\text{dv}}{\text{dx}} = \text{g(x)}$ then $\text{v} = \int \text{g(x)dx}$

$$\int f(x)g(x)dx = f(x)\int g(x)dx - \int [f'(x)\int g(x)dx]dx$$

Thus integral of the product of two functions

= 1^{st} function × integral of the 2^{nd} – integral of (differential of 1^{st} × integral of 2^{nd}).

It is clear from the formula that it is helpful only when we know integral of at least one of the two given functions.

The following examples will illustrate how to apply this rule.

Example-5:

Evaluate $\int xe^x dx$

Solution:

$$\int xe^x dx = x\int e^x dx - \int \{\frac{d}{dx}(x)\int e^x dx\}dx$$
$$= xe^x - \int e^x dx$$
$$= xe^x - e^x + c$$

Example-6:

Evaluate $\int \log x \, dx$

Solution:

$$\int \log x \, dx = \int \log x.1.\, dx$$
$$= \log x\int 1.dx - \int \{\frac{d}{dx}(\log x)\int 1.dx\}dx$$
$$= \log x.x - \int \frac{1}{x}.xdx$$
$$= x\log x - \int dx$$
$$= x\log x - x + c\,.$$

Example 7:

Evaluate $\int x^2 \log x \, dx$

Solution:

$$\int x^2 \log x \, dx = \log x\int x^2 dx - \int \{\frac{d}{dx}(\log x)\int x^2 dx\}dx$$
$$= \log x.\frac{x^3}{3} - \int \frac{1}{x}.\frac{x^3}{3}.dx$$
$$= \frac{x^3}{3}\log x - \int \frac{x^2}{3}dx$$
$$= \frac{x^3}{3}\log x - \frac{x^3}{9} + c$$

Integration by Successive Reduction

Any formula expressing a given integral in terms of another that is simpler than it, is called a reduction formula for the given integral. In practice, however, the reduction formula for a given integral means that the integral belongs to class of integrals such that it can be expressed in terms of one or more integrals or lower orders belonging to the same class; by successive application of the formula, we arrive at integrals which can be easily integrated and hence the given integral can be evaluated.

Any formula expressing a given integral in terms of another that is simpler than it, is called a reduction formula for the given integral.

Example-8:

Evaluate $\int x^3 e^{3x} \, dx$

Solution:

$$\int x^3 e^{3x}\, dx = x^3 \int e^{3x} dx - \int \{\frac{d}{dx}(x^3) \int e^{3x} dx\} dx$$

$$= \frac{x^3 e^{3x}}{3} - \int x^2 e^{3x} dx$$

$$= \frac{x^3 e^{3x}}{3} - \left[\frac{x^2 e^{3x}}{3} - \frac{2}{3}\int x e^{3x} dx\right]$$

$$= \frac{x^3 e^{3x}}{3} - \frac{x^2 e^{3x}}{3} + \frac{2}{3}\left[\frac{x e^{3x}}{3} - \frac{1}{3}\int e^{3x} dx\right]$$

$$= \frac{x^3 e^{3x}}{3} - \frac{x^2 e^{3x}}{3} + \frac{2}{9} x e^{3x} - \frac{2}{27} e^{3x} + c$$

Integration by Partial Fraction

Many rational functions exist which cannot be integrated by the rules of integration.

Rational functions have the form of a quotient of two polynomials. Many rational functions exist which cannot be integrated by the rules of integration presented earlier. When these occur, one possibility is that the rational function can be restated in an equivalent form consisting of more elementary functions and then each of the component fractions can be easily integrated separately. The following examples illustrate the decomposition of a rational function into equivalent partial fractions.

Example-9:

Evaluate $\int \frac{x+3}{x^2+3x+2} dx$

Solution:

$$\int \frac{x+3}{x^2+3x+2} dx = \int \frac{x+3}{(x+1)(x+2)} dx$$

$$= \int \left(\frac{2}{x+1} - \frac{1}{x+2}\right) dx$$

$$= 2 \log (x+1) - \log (x+2) + c$$

Example-10:

Evaluate $\int \frac{3x+1}{x+1} dx$

Solution:

$$\int \frac{3x+1}{x+1} dx = \int (3 - \frac{2}{x+1}) dx$$

$$= \int 3\, dx - \int \frac{2}{x+1} dx$$

$$= 3x - 2 \log (x+1) + c$$

Example-11:

Evaluate $\int \frac{5x+8}{x^2+4x+4}\,dx$

Solution:

$$\int \frac{5x+8}{x^2+4x+4}\,dx = \int\left(\frac{5}{x+2}-\frac{2}{(x+2)^2}\right)dx$$

$$= \int \frac{5}{x+2}\,dx - \int \frac{2}{(x+2)^2}\,dx$$

$$= 5\log(x+2)+\frac{2}{x+2}+c$$

Example 12:

Evaluate $\int \frac{x^3-2x}{x-1}\,dx$

Solution:

$$\int \frac{x^3-2x}{x-1}\,dx = \int (x^2+x-1-\frac{1}{x-1})\,dx$$

$$= \int (x^2+x-1)dx - \int \frac{1}{x-1}dx$$

$$= \frac{x^3}{3}+\frac{x^2}{2}-x-\log(x-1)+c$$

Example 13:

Evaluate $\int \frac{2x^2-1}{x^3+x^2}\,dx$

Solution:

$$\int \frac{2x^2-1}{x^3+x^2}\,dx = \int\left(\frac{x-1}{x^2}+\frac{1}{x+1}\right)\mathrm{dx}$$

$$= \int \frac{x-1}{x^2}dx + \int \frac{1}{x+1}dx$$

$$= \log x + \frac{1}{x} + \log(x+1)+c$$

Questions for Review

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. Integrate the following functions w. r. t. x

(i) $\int \frac{dx}{x^2(a+bx)^2}$

(ii) $\int \frac{8x^2}{(x^3+2)^3}dx$

(iii) $\int x \log x dx$

(iv) $\int x^3 e^x dx$

(v) $\int \frac{x^2-2}{(x+1)(x^2+1)}dx$

Lesson-3: Definite Integral

After studying this lesson, you should be able to:

- Describe the concept of definite integral;
- Evaluate definite integrals.

Introduction

In Geometry and other application areas of integral calculus, it becomes necessary to find the difference in the values of an integral *f(x)* for two assigned values of the independent variable *x*, say, *a, b, (a < b),* where *a* and *b* are two real numbers. The difference is called the definite integral of *f(x)* over the domain (*a, b*) and is denoted by

$$\int_a^b f(x)dx............(i)$$

If *g(x)* is an integral of *f(x)*, then we can write,

$$\int_a^b f(x)dx = \left[g(x)\right]_a^b = g(b) - g(a)..............(ii)$$

Here $\int_a^b f(x)dx$ is called the definite integral, as the constant of integration does not appear in it. If we consider [*g(x)* + *c*] instead of *g(x)*, we have

$$\int_a^b f(x)dx = \left[g(x)+c\right]_a^b = g(b) + c - g(a) - c = g(b) - g(a)........(iii)$$

Thus, from (ii) or (iii), we get a specific numerical value, free of the variable *x* as well as the arbitrary constant *c*. This value is called the definite integral of *f(x)* from *a* to *b*. We refer to *a* as the lower limit of integration and to *b* as the upper limit of integration.

Properties of Definite Integral

Some important properties of definite integral are given below:

$$(i)\int_a^b f(x)dx = \int_b^a f(z)dz$$

$$(ii)\int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$(iii)\int_a^a f(x)dx = 0$$

$$(iv)\int_a^c f(x)dx = \int_a^b f(x) + \int_b^c f(x)dx$$

$$(v)\int_a^b cf(x)dx = c\int_a^b f(x)dx$$

$$(vi)\int_0^a f(x)dx = \int_0^a f(a-x)dx$$

$$(vii)\int_0^{na} f(x)dx = n\int_0^a f(x)dx \qquad if \quad f(a+x) = f(x)$$

$$(viii)\int_0^{2a} f(x)dx = 2\int_0^a f(x)dx \qquad if \quad f(2a-x) = f(x)$$

$$(ix)\int_{-a}^{+a} f(x)dx = 2\int_a^a f(x)dx$$

Illustrative Examples:

Example-1:

Evaluate $\int_a^b c\,dx$

Solution:

$$\int_a^b c\,dx = c\int_a^b dx = c[x]_a^b = c\,(b-a)$$

Example-2:

Evaluate $\int_1^4 x^2\,dx$

Solution:

$$\int_1^4 x^2\,dx = \left[\frac{x^3}{3}\right]_1^4 = 21$$

Example-3:

Evaluate $\int_1^9 \frac{dx}{\sqrt{x}}\,dx$

Solution: $\int_1^9 \frac{dx}{\sqrt{x}}\,dx = \int_1^9 x^{-\frac{1}{2}}\,dx = 2\left[x^{\frac{1}{2}}\right]_1^9 = 4$

Example-4:

Evaluate $\int_1^4 (2x^2 - 4x + 5)\,dx$

Solution:

$$\int_1^4 (2x^2 - 4x + 5)\,dx = \left[\frac{2x^3}{3} - 2x^2 + 5x\right]_1^4 = 27$$

Example-5:

Evaluate $\int_{2}^{4} \frac{x\,dx}{x^2-1}$

Solution:

Let $x^2 - 1 = z$

$2xdx = dz$

when $x = 2$, then $z = 3$

when $x = 4$, then $z = 15$

$$\int_{2}^{4} \frac{x\,dx}{x^2-1} = \int_{3}^{15} \frac{dz}{2z} = \frac{1}{2}\left[\log z\right]_{3}^{15} = \frac{1}{2}(\log 15 - \log 3)$$

Example-6:

Evaluate $\int_{0}^{4} x\sqrt{x^2+9}\,dx$

Solution:

Let $x^2 + 9 = z$

$2xdx = dz$

when $x = 0$, then $z = 9$

when $x = 4$, then $z = 25$

$$\int_{0}^{4} x\sqrt{x^2+9}\,dx = \frac{1}{2}\int_{9}^{25} \sqrt{z}\,dz = \frac{1}{2}\left[\frac{z^{\frac{3}{2}}}{\frac{3}{2}}\right]_{9}^{25} = \frac{98}{3}$$

Example-7:

Evaluate $\int_{0}^{\pi} \frac{xdx}{1+\sin x}\,dx$

Solution:

Let $I = \int_{0}^{\pi} \frac{xdx}{1+\sin x}\,dx = \int_{0}^{\pi} \frac{(\pi - x)dx}{1+\sin(\pi - x)}\,dx = \int_{0}^{\pi} \frac{(\pi - x)dx}{1+\sin x}\,dx$

$$= \int_{0}^{\pi} \frac{\pi dx}{1+\sin x}\,dx - \int_{0}^{\pi} \frac{xdx}{1+\sin x}\,dx$$

$$2I = \pi\int_{0}^{\pi} \frac{dx}{1+\sin x}\,dx = \pi\int_{0}^{\pi} \frac{1-\sin x}{\cos^2 x}\,dx =$$

$$\pi\left[\int_{0}^{\pi} \sec^2 x\,dx - \int_{0}^{\pi} \tan x \sec x dx\right]$$

$$= \pi\left[\tan x - \sec x\right]_{0}^{\pi}$$

$$= 2\pi$$

$\therefore I = \pi$.

Proper and Improper Integrals

An integral is said to be proper integral when it is bounded and the range of integration is finite.

Proper Integrals: An integral is said to be proper integral when it is bounded and the range of integration is finite. For example, $\int_0^2 f(x)dx$, $\int_1^2 f(x)dx$ etc.

Improper Integrals: When the range of integration is finite but the integrand is unbounded for some values in the range of integration, then it is called the improper integral of first kind. e.g. $\int_1^2 \frac{dx}{(1-x)(2-x)}$, $\int_0^1 \frac{dx}{x^2}$ etc.

When the range of integration is infinite but the integrand is bound, then it is called improper integrals of second kind. e.g.,$\int_a^\infty f(x)dx$, $\int_{-\infty}^b f(x)dx$, $\int_{-\infty}^\infty f(x)dx$ etc.

These types of improper integrals are determined as if

$$\int_a^\infty f(x)dx = \lim_{w\to\infty}\int_a^w f(x)dx.$$

When the limit exists, the integral is said to be convergent to that limit and when the limit does not exist, the integral is said to be divergent to that limit.

When the limit exists, the integral is said to be convergent to that limit and when the limit does not exist, the integral is said to be divergent to that limit. If, however, the limit does not converge or diverge then it is said to be oscillatory.

Example-8:

Determine whether the improper integral $\int_0^\infty \frac{e^x}{2}dx$ is convergent or divergent.

Solution:

$$\int_0^\infty \frac{e^x}{2}dx = \lim_{a\to\infty}\int_0^a \frac{e^x}{2}dx \quad = \lim_{a\to\infty}\frac{1}{2}(e^a - 1) = \infty$$

Thus the given improper integral is divergent.

Example-9:

Determine whether the improper integral $\int_{-\infty}^0 e^x dx$ is convergent or divergent.

Solution:

$$\int_{-\infty}^{0} e^{x} dx = \lim_{a \to -\infty} \int_{a}^{0} e^{x} dx = \lim_{a \to -\infty} (e^{0} - e^{a})$$

$$= \lim_{a \to -\infty} (1 - e^{a}) = 1 - 0 = 1$$

Thus the given improper integral is convergent and its value is 1.

Multiple Integrals

Integration of a function in one variable generates an area (a surface) from a line. A function in two variables generates a volume from a surface. Because a function in two variables describes a surface with different curvature in each direction, both variables are responsible for generating the appropriate volume. To find the volume, the integral must be taken with respect to both variables.

A function in two variables generates a volume from a surface.

Example-10:

Find the value of $\int_{0}^{1} \int_{0}^{2} (x + 2) dy dx$

Solution:

$$\int_{0}^{1} \int_{0}^{2} (x + 2) dy dx = \int_{0}^{1} \left[\int_{0}^{2} (x + 2) dy \right] dx$$

$$= \int_{0}^{1} [xy + 2y]_{0}^{2} dx$$

$$= \int_{0}^{1} [2x + 4] dx$$

$$= [x^{2} + 4x]_{0}^{1}$$

$$= 5$$

Example-11:

Find the value of $\int_{2}^{3} \int_{1}^{2} \int_{2}^{5} xydzdydx$

Solution:

$$\int_{2}^{3} \int_{1}^{2} \int_{2}^{5} xydzdydx = \int_{2}^{3} \int_{1}^{2} \left[\int_{2}^{5} xydz \right] dydx$$

$$= \int_{2}^{3}\int_{1}^{2} [\, xyz\,]_{2}^{5}\, dydx$$

$$= 3\int_{2}^{3} [\int_{1}^{2} xy\, dy\,]\, dx$$

$$= 3\int_{2}^{3} [\frac{1}{2} xy^{2}\,]_{1}^{2}\, dx$$

$$= 3.\frac{3}{2}.\int_{2}^{3} x\, dx$$

$$= \frac{45}{4}$$

Questions for Review:

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. Evaluate the following integrals:

   (i) $\int_a^b e^x \, dx$

   (ii) $\int_0^1 x^3 \sqrt{(1+3x^4)} \, dx$

   (iii) $\int_a^b \frac{\log x}{x} \, dx$

   (iv) $\int_0^\infty \frac{dx}{1+x^2}$

   (v) $\int_0^{\pi/2} \log \sin x \, dx$

   (vi) $\int_0^2 \int_0^x (x^2 + y^2) \, dy dx$

   (vii) $\int_1^2 \int_0^1 \int_{-1}^1 (x^2 + y^2 + z^2) \, dx dy dz$

Lesson-4: Applications of Integration in Business

After studying this lesson, you should be able to:

- Express the importance of integration in Business and Economics;
- Apply integration in different types of business decisions.

Introduction

The knowledge of integration is widely used in business and economics. For example, net investment *I* is defined as the rate of change in capital stock formation *K* over time *t*. If the process of capital formation is continuous over time, $I(t) = \frac{dK(t)}{dt} = K'(t)$. From the rate of investment, the level of capital stock can be estimated. Capital stock is the integral with respect to time of net investment. Similarly the integral can be used to estimate total cost from the marginal cost. Since marginal cost is the change in total cost from an incremental change in output and only variable costs change with the level of output. Economic analysis that traces the time path of variables or attempts to determine whether variables will converge towards equilibrium over time is called dynamics. Thus, we can use integration in many business decision making processes. In this lesson, we discuss about a few sample applications of integration. The following examples illustrate sample applications of integral calculus.

Marginal cost is the change in total cost from an incremental change in output and only variable costs change with the level of output.

Illustrative Examples:

Example-1:

Find the area bounded by the curve $y = x^2$, the *x*-axis and the lines $x = 1$ and $x = 3$.

Solution:

We know that, Area = $\int_a^b y\,dx = \int_1^3 x^2\,dx = \left[\frac{x^3}{3}\right]_1^3 = \frac{26}{3}$

Example-2:

The marginal cost function of a product is given by $\frac{dC}{dq} = 100 - 10q + 0.1q^2$, where *q* is the output. Find the total and average cost functions of the firm assuming that its fixed cost is \$1500.

Solution:

Given that $\frac{dC}{dq} = 100 - 10q + 0.1q^2$

Integrating this with respect to *q*, we get,

C = $\int (100 - 10q + 0.1q^2)\,dq$

C = $100q - 5q^2 + \frac{0.10}{3}q^3 + K$

Now the fixed cost is 1500; i.e., when $q = 0$, C = 1500.
$\therefore$ K = 1500

Hence the total cost function is, C = $100q - 5q^2 + \frac{0.10}{3}q^3 + 1500$

And the average cost is $\frac{C}{q} = 100 - 5q + \frac{0.1}{3}q^2 + \frac{1500}{q}$

Example-3:

The marginal revenue $R'(x) = 25 - 8x + 6x^2 + 4x^3$. Find the revenue function.

Solution:

The revenue function is

$$R(x) = \int R'(x)\,dx = \int (25 - 8x + 6x^2 + 4x^3)\,dx$$

$$= 25x - 4x^2 + 2x^3 + x^4 + K.$$

We know that when no product is sold then the revenue is zero. i.e., when $x = 0$, R = 0.

Thus, K = 0.

Thus the revenue function is $R(x) = 25x - 4x^2 + 2x^3 + x^4$.

Example-4:

The marginal cost function for a certain commodity is $MC = 3q^2 - 4q + 5$. Find the cost of producing the 11th through the 15th units, inclusive.

Solution: $\int_{11-1}^{15} (3q^2 - 4q + 5)dq$

$$= \int_{10}^{15} (3q^2 - 4q + 5)dq$$

$$= \left[q^3 - 2q^2 + 5q\right]_{10}^{15}$$

$= 2150$

Example-5:

A Company determines that the marginal cost of producing x units of a particular commodity during a one-day operation is $MC = 16x - 1591$, where the production cost is in dollar. The selling price of a commodity is fixed at \$9 per unit and the fixed cost is \$1800 per day.

(i) Find the cost function.
(ii) Find the revenue function.
(iii) Find the profit function.
(iv) Find the maximum profit that can be obtained in a one-day operation.

Solution: Given that $MC = 16x - 1591$

FC = 1800

P = 9

(i) Cost function, TC = $\int MC\, dx$

$$= \int (16x - 1591)\, dx$$

$$= 8x^2 - 1591x + c$$

$$= 8x^2 - 1591x + 1800$$

(ii) Revenue = $P \times x = 9x$

(iii) Profit = TR – TC

$$= 9x - (8x^2 - 1591x + 1800)$$

$$= 1600x - 8x^2 - 1800$$

(iv) Profit $y = 1600x - 8x^2 - 1800$

$$\frac{dy}{dx} = 1600 - 16x$$

For maximum or minimum, $\frac{dy}{dx} = 0$

$$1600 - 16x = 0$$

$$x = 100$$

Again, $\frac{d^2y}{dx^2} = -16$

Hence the required profit maximizing sales volume is $x = 100$.

Required maximum profit $y = 1600x - 8x^2 - 1800$

$$= 1600(100) - 8(100)^2 - 1800$$

= \$78200.

Example-6:

After an advertising campaign a product has sales rate *f(t)* given by $f(t) = 1000\, e^{-0.5t}$ where *t* is the number of months since the close of the campaign.

(i) Find the total cumulative sales after 3 months.

(ii) Find the sales during the fourth month.

(iii) Find the total sale as a result of campaign.

Solution:

Let *F(t)* is the total sale after *t* months since the close of the campaign. The sale rate is *f(t)*.

$$F(t) = \int_0^t f(t)dt$$

(i) The total cumulative sales after 3 months, $F(3) = \int_0^3 f(t)dt$

$$\int_0^3 f(t)dt = \int_0^3 1000e^{-0.5t}dt$$

$= \frac{1000}{-0.5}\left[e^{-0.5t}\right]_0^3$

= –2000 [$e^{-1.5}$ –1]

= –2000 (0.2231 – 1)

= 1554 units.

(ii) Sales during the fourth month = $\int_3^4 1000e^{-0.5t}dt$

$\int_3^4 1000e^{-0.5t}dt = \frac{1000}{-0.5}\left[e^{-0.5t}\right]_3^4$

= –2000 [$e^{-2.0}$ – $e^{-1.5}$]

= –2000 (0.1353 – 2231)

= 175.6 units.

(iii) Total sales as a result of campaign = $\int_0^\infty 1000e^{-0.5t}dt$

$\int_0^\infty 1000e^{-0.5t}dt = \frac{1000}{-0.5}\left[e^{-0.5t}\right]_0^\infty$

= –2000 (0 – 1)

= 2000 units.

Example-7:

If \$500 is deposited each year in a saving account pays 5.5 % per annum compounded continuously, how much is in the account after 4 years?

Solution:

Given that, payment per year, $P = 500$

Rate of interest, $r = 0.055$

Time, $t = 4$

Amount of the future value A = $\int_0^t P\,e^{rt}dt$

$= \int_0^4 500e^{0.055t}dt$

$= \frac{500}{0.055}\left[e^{0.055t}\right]_0^4$

= 9090.91 [$e^{0.22}$ – e^0]

= 9090.91(0.246076)

= \$2237.

Example-8:

If the marginal revenue and the marginal cost for an output x of a commodity are given as $MR = 5 - 4x + 3x^2$ and $MC = 3 + 2x$, and if the fixed cost is zero, find the profit function and the profit when the output $x = 4$.

Solution:

Given that, $MR = 5 - 4x + 3x^2$

$MC = 3 + 2x$

Profit = $TR - TC$

$$= \int MR dx - \int MC dx$$

$$= \int (5 - 4x + 3x^2) dx - \int (3 + 2x) dx$$

$$= (5x - 2x^2 + x^3 + c_1) - (3x + x^2 + c_2)$$

Since the fixed cost is zero so that $c_2 = 0$; for $x = 0$, total revenue = 0

Profit = $2x - 3x^2 + x^3$

When $x = 4$, the profit = $(2 \times 4) - 3(4)^2 + (4)^3 = 24$.

Example-9:

Find the total cost function if it is known that the cost of zero output is c and that marginal cost of output x is $ax + b$.

Solution:

We are given that, Marginal cost (MC) = $ax + b$.

$$\frac{dTC}{dx} = ax + b$$

$$TC = \int (ax + b) dx$$

$$TC = \frac{ax^2}{2} + bx + K$$

When $x = 0$, $TC = c$, so, $K = c$.

Hence the total cost function is given by, TC = $\frac{ax^2}{2} + bx + c$

Example-10:

Let the rate of net investment is given by $I(t) = 9t^{1/2}$, find the level of capital formation in (i) 8 years (ii) for the fifth through the eighth years.

Solution:

(i) $K = \int_0^8 9t^{1/2} dt = 6t^{3/2}\Big|_0^8 = 135.76$

(ii) $K = \int_4^8 9t^{1/2} dt = 6t^{3/2}\Big|_4^8 = 87.76$

Questions for Review:

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. Marginal cost is given by $MC = 25 + 30Q - 9Q^2$. Fixed cost is 55. Find the (i) total cost (ii) average cost, and (iii) variable cost functions.
2. Marginal revenue is given by $MR = 60 - 2Q - 2Q^2$. Find the total revenue function and the demand function.
3. The rate of net investment is $I = 40t^{3/5}$ and capital stock at t = 0 is 75. Find the capital function *K*.

Matrix Algebra

11

This unit is designed to introduce the learners to the basic concepts associated with matrix algebra. The learners will learn about different types of matrices, operations of matrices, determinant and matrix inversion. This unit also discusses the procedure of determining the solution of the system of linear equations by using inverse matrix method, Gaussian method and Cramer's' Rule. Some relevant business and economic applications of matrix algebra are also provided in this unit for clear and better understanding of the learners.

Blank Page

Lesson-1: Matrix: An Introduction

After studying this lesson, you should be able to:

- State the nature of a matrix;
- Explain matrix representation of data.
- Define different types of matrices.

Introduction

J. J. Sylvester was the first to use the word 'matrix' in 1850 and later on in 1858 Arthur Cayley developed the theory of matrices in a systematic way. Matrix is a powerful tool of modern mathematics and its study is becoming important day by day due to its wide applications in every branch of knowledge. Matrix arithmetic is basic to many of the tools of managerial decision analysis. It has an important role in modern techniques for quantitative analysis of business and economic decisions. The tool has also become quite significant in the functional business and economic areas of accounting, production, finance and marketing.

Matrix arithmetic is basic to many of the tools of managerial decision analysis.

Matrix

Whenever one is dealing with data, there should be concern for organizing them in such a way that they are meaningful and can be readily identified. Summarizing data in a tabular form can serve this function. A matrix is a common device for summarizing and displaying numbers or data. Thus, a matrix is a rectangular array of elements and has no numerical value. The elements may be numbers, parameters or variables. The elements in horizontal lines are called rows, and the elements in vertical lines are called columns.

A matrix is a rectangular array of elements and has no numerical value.

A matrix is characterized further by its dimension. The dimension or order indicates the number of rows and the number of columns contained within the matrix. If a matrix has *m* rows and *n* columns, it is said to have dimension ($m \times n$), which is read as: *m* by *n*.

Example: $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

Types of Matrices:

Row Matrix: The matrix with only one row is called a row matrix or row vector.

For example: $A = (2 \quad 3 \quad 4)$.

Column Matrix: The matrix with only one column is called a column matrix or column vector.

For example: $A = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$

Row matrix and column matrix are usually called as row vector and column vector respectively.

Square Matrix: If the number of rows and the number of columns of a matrix are equal then the matrix is of order $n \times n$ and is called a square matrix of order *n*.

For example: $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

Rectangular Matrix: If the number of rows and the number of columns of a matrix are not equal then the matrix is called a rectangular matrix.

For example: $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$

Singular matrix: A square matrix A is said to be singular if the determinant formed by its elements equal to zero.

For example: Let $A = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}$.

Determinant of $A = |A| = (2 \times 2) - (4 \times 1) = 0$.

Hence *A* is a singular matrix.

Non-singular Matrix:

A square matrix *A* is said to be non-singular if the determinant formed by its elements is non-zero.

For example: $A = \begin{pmatrix} 5 & 3 \\ 2 & 4 \end{pmatrix}$

$|A| = (5 \times 4) - (3 \times 2) = 20 - 6 = 14$.

Hence *A* is a non-singular matrix.

Null or Zero Matrix: The matrix with all of its elements equal to zero is called a null matrix or zero matrix.

For example: $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Diagonal Matrix: A matrix whose all elements are zero except those in the principal diagonal is called a diagonal matrix.

For example: $A = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}$

Scalar Matrix: A diagonal matrix, whose diagonal elements are equal, is called a scalar matrix.

For example: $A = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$

Sub-Matrix: A matrix that is obtained from a given matrix by deleting any number of rows and any number of columns is called a sub-matrix of the given matrix.

For example: $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ is a sub-matrix of $B = \begin{pmatrix} 5 & 3 & 2 \\ 1 & 1 & 2 \\ 7 & 3 & 4 \end{pmatrix}$

Unit matrix or Identity matrix: A matrix with every element in the principal diagonal equals to one and all other elements equal to zero is called a unit matrix. A unit matrix is a square matrix. It is denoted by *I*.

For example: $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Equal Matrix: Two matrices *A* and *B* are said to be equal if their corresponding elements are equal.

For example: Let $A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$ then $A = B$

Transpose of a Matrix: If the columns of a given matrix *A* are changed into rows or the rows are changed into columns, the matrix thus formed is called the transpose of the matrix *A* and it is generally denoted by A^T.

For example: Let $\mathrm{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ then $\mathrm{A}^\mathrm{T} = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$

Symmetric Matrix: A square matrix *A* is called symmetric if it be same as its transpose so that $A = A^T$.

For Example: Let $\mathrm{A} = \begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix}$ then $\mathrm{A}^\mathrm{T} = \begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix}$

i.e., $A = A^T$, so *A* is a symmetric matrix.

Skew-Symmetric Matrix: A square matrix *A* is called skew-symmetric if $A^T = -A$.

For example: Let $\mathrm{A} = \begin{pmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{pmatrix}$

then $A^T = \mathrm{A}^\mathrm{T} = \begin{pmatrix} 0 & -h & -g \\ h & 0 & -f \\ g & f & 0 \end{pmatrix} = -A$

i.e., $A^T = -A$, hence *A* is a skew-symmetric matrix.

Involuntary Matrix: A square matrix *A* is called involuntary matrix provided it satisfies the relation $A^2 = I$, where *I* is the identity matrix.

For example: $A = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$

Idempotent Matrix: A square matrix *A* is called idempotent matrix provided it satisfies the relation $A^2 = A$.

Example: $A = \begin{pmatrix} 2 & -2 & 4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{pmatrix}$

Nilpotent Matrix: A square matrix *A* is called nilpotent matrix of order ***m*** provided it satisfies the relation $A^m = 0$ and $A^{m-1} \neq 0$, where *m* is a positive integer and *0* is the null matrix.

For example: $A = \begin{pmatrix} 1 & 2 & 5 \\ 2 & 4 & 10 \\ -1 & -2 & -5 \end{pmatrix}$ since $A \neq 0, A^2 = 0$

Complex Conjugate of a Matrix: It is a matrix obtained by replacing all its elements by their respective complex conjugates.

For example: If $A = \begin{pmatrix} 2 & +3i & 5 \\ 3 & -3i & 7 \end{pmatrix}$ then $\overline{A} = \begin{pmatrix} 2 & -3i & 5 \\ 3 & +3i & 7 \end{pmatrix}$

Hermitian Matrix: A matrix having complex elements of a square matrix *A* is a Hermitian matrix. If $(A)' = A$, then *A* is called Hermitian matrix.

Skew-Hermitian Matrix: A matrix having complex elements for matrix $A.(A) = -A.A$ is skew hermitian matrix.

Co-factor Matrix

A matrix, which is formed by the co-factors of the corresponding elements, is called co-factor matrix and is denoted by A^C.

For example: If a matrix, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

then, the co-factor matrix, $A^C = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}$

Adjoint Matrix:

The Adjoint matrix is the transpose of the co-factor matrix, that is $adjA = A_j = (cof\ A)^T$

Orthogonal Matrix: A square matrix *A* is called an orthogonal matrix if $AA^T = A^T A = I$, where *I* is an identity matrix and A^T is the transpose matrix of *A*.

Questions for Review:

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. What do you understand by matrix?
2. Why matrix algebra is so important in business and economics? Explain.
3. Discuss the various types of matrices.
4. In an examination, 20 students from college A, 30 students from college B and 40 students from college C appeared. Only 15 students from each college could get through the examination. Out of them 10 students from college A, 5 students from college B and 10 students from college C secured full marks. Write down the above data in matrix form.

Lesson-2: Matrix Operations

After studying this lesson, you should be able to:

- Express the concept of matrix operations;
- Add or subtract given matrices;
- Multiply given matrices.

Introduction

The operations of matrices are addition, subtraction, multiplication and division of which addition and multiplication are the main operations. In this lesson we will discuss some of the operations of matrix algebra.

Matrix Addition

Two matrices of the same dimensions are said to be conformable for addition.

Two matrices of the same dimensions are said to be conformable for addition. The addition is performed by adding corresponding elements from the two matrices and entering the result in the same row-column position of a new matrix.

If *A* and *B* are two matrices, each of size $m \times n$ then the sum of *A* and *B* is the $m \times n$ matrix *C* whose elements are $C_{ij} = A_{ij} + B_{ij}$; i = 1, 2, 3… m and j = 1, 2, 3, ….n.

Properties of Matrix Addition:

- *Commutative law*: Matrix addition is commutative. If *A* and *B* are two matrices of same order $m \times n$, then $A + B = B + A$.
- *Associative law*: Matrix addition is associative. If *A*, *B* and *C* are three matrices of same order $m \times n$, then $A + (B + C) = (A + B) + C$.
- *Distributive law*: If *A* and *B* are two matrices of same order $m \times n$, and *K* is any scalar, then $K(A + B) = KA + KB$.
- *Existence of additive identity*: If *O* denotes null matrix of the same order as that of *A*, then $A + O = A = O + A$.
- *Existence of an additive inverse*: If *A* be any given $m \times n$ matrix and there exists another $m \times n$ matrix *B* such that $A + B = O = B + A$; where *O* be the $m \times n$ null matrix.
- *Cancellation law*: If *A*, *B* and *C* are three matrices of same order$(m \times n)$, then $A + C = B + C \Rightarrow A = B$.

Example-1:

Find the sums $A + B$ of the following matrices

$$A = \begin{pmatrix} 8 & 9 \\ 12 & 7 \end{pmatrix} \text{ and } B = \begin{pmatrix} 13 & 4 \\ 2 & 6 \end{pmatrix}$$

Solution:

$$A + B = \begin{pmatrix} 8+13 & 9+4 \\ 12+2 & 7+6 \end{pmatrix} = \begin{pmatrix} 21 & 13 \\ 14 & 13 \end{pmatrix}$$

Matrix Subtraction

The subtraction of two matrices is possible only when they are of the same order. Such matrices are said to be conformable for subtraction. The subtraction is performed by subtracting corresponding elements of the two matrices and entering the result in the same row-column position of a new matrix.

The subtraction of two matrices is possible only when they are of the same order.

If A and B are two matrices, each of size $m \times n$ then the subtraction of A and B is the $m \times n$ matrix C whose elements are $C_{ij} = A_{ij} - B_{ij}$; i = 1, 2, 3… m and j = 1, 2, 3, ….n.

Example-2:

Find the difference $A - B$ of the following matrices

$$A = \begin{pmatrix} 3 & 7 & 11 \\ 12 & 9 & 2 \end{pmatrix} \text{ and } B = \begin{pmatrix} 6 & 8 & 1 \\ 9 & 5 & 8 \end{pmatrix}$$

Solution:

$$A - B = \begin{pmatrix} 3-6 & 7-8 & 11-1 \\ 12-9 & 9-5 & 2-8 \end{pmatrix} = \begin{pmatrix} -3 & -1 & 10 \\ 3 & 4 & -6 \end{pmatrix}$$

Scalar Multiplication of a Matrix

A matrix can be multiplied by a constant by multiplying each component in the matrix by the constant. The result is a new matrix of the same dimensions as the original matrix.

A matrix can be multiplied by a constant by multiplying each component in the matrix by the constant.

If K is any real number and $A = [a_{ij}]$ is an $m \times n$ matrix, then the product KA is defined to be the matrix whose components are given by K times the corresponding component of A, i.e., $KA = [Ka_{ij}]$

Laws of scalar multiplication:

(i) $K(A+B) = KA + KB$

(ii) $(K_1 + K_2)A = K_1A + K_2A$

(iii) $IA = A$

(iv) $(K_1K_2)A = K_1(K_2A)$.

Example-3:

If $A = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix}$, Find $5A$.

Solution:

$$5A = 5\begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 0 & 5 \\ 10 & 5 & 10 \\ 15 & 10 & 5 \end{pmatrix}$$

Multiplication of Matrices

If the number of columns of the first matrix is equal to the number of rows of the second matrix, such matrices are said to be conformable for multiplication.

If the number of columns of the first matrix is equal to the number of rows of the second matrix, such matrices are said to be conformable for multiplication. Let A be a matrix of order $m \times p$ and B be a matrix of order $p \times n$. Then the product AB is defined to be a matrix C of order $m \times n$.

Properties of Matrix Multiplication

- *Associative law*: Multiplication of matrices is associative i.e. $A(BC) = (AB)C$.
- *Distributive law*: Multiplication of matrices is distributive with respect to matrix addition i.e. $A(B+C) = AB + AC$.
- *Multiplication of a matrix by a null matrix:* If A is $n \times m$ and O is $m \times n$ matrices, then $AO = O = OA$.
- *Multiplication of a matrix by a unit matrix:* If A is a square matrix of order $n \times n$ and I is the unit matrix of same order, then $IA = A = AI$.
- *Multiplication of matrix by itself:* If A is a square matrix then $A.A = A^2$.

Example-4:

Find AB, where $A = [9 \quad 11 \quad 3]$ and $B = \begin{pmatrix} 2 \\ 6 \\ 7 \end{pmatrix}$

Solution:

The matrices A and B are conformable for multiplication. The dimensions of A and B are 1×3 and 3×1 respectively, i.e., the product matrix AB will be 1×1 and a scalar, derived by multiplying each element of the row vector by its corresponding element in the column vector and then summing the products.

$$AB = [(9 \times 2) + (11 \times 6) + (3 \times 7)] = 18 + 66 + 21 = 105.$$

Example-5:

If $A = \begin{pmatrix} 2 & 3 & 1 \\ 0 & -1 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix}$.

Find (i) $3A - 4B$

(ii) $2A - 3B$

Solution:

(i) $3A - 4B = 3\begin{pmatrix} 2 & 3 & 1 \\ 0 & -1 & 5 \end{pmatrix} - 4\begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix}$

$$= \begin{pmatrix} 6 & 9 & 3 \\ 0 & -3 & 15 \end{pmatrix} - \begin{pmatrix} 4 & 8 & -4 \\ 0 & -4 & 12 \end{pmatrix}$$

$$= \begin{pmatrix} 6-4 & 9-8 & 3-(-4) \\ 0-0 & -3-(-4) & 15-12 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 & 7 \\ 0 & 1 & 3 \end{pmatrix}$$

(ii) $2A - 3B = 2\begin{pmatrix} 2 & 3 & 1 \\ 0 & -1 & 5 \end{pmatrix} - 3\begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \end{pmatrix}$

$$= \begin{pmatrix} 4 & 6 & 2 \\ 0 & -2 & 10 \end{pmatrix} - \begin{pmatrix} 3 & 6 & -3 \\ 0 & -3 & 9 \end{pmatrix}$$

$$= \begin{pmatrix} 4-3 & 6-6 & 2-(-3) \\ 0-0 & -2-(-3) & 10-9 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 1 \end{pmatrix}$$

Example-6:

If $A = \begin{pmatrix} 3 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 4 \\ 2 & 2 \\ 1 & 0 \end{pmatrix}$

then find *AB*. Whether *BA* exists? Give reason.

Solution:

$$AB = \begin{pmatrix} 3 & 1 & 2 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{pmatrix} \times \begin{pmatrix} 1 & 4 \\ 2 & 2 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 3.1+1.2+2.1 & 3.4+1.2+2.0 \\ 0.1+1.2+1.1 & 0.4+1.2+1.0 \\ 1.1+2.2+0.1 & 1.4+2.2+0.0 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & 14 \\ 3 & 2 \\ 5 & 8 \end{pmatrix}$$

Here *A* is a matrix of order 3×3 and *B* is a matrix of order 3×2. Hence *BA* does not exist as number of columns in *B* is not equal to the number of rows in *A*.

Example-7:

If $A = \begin{pmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{pmatrix}$.

Find AB and show that $AB \neq BA$

Solution:

$$AB = \begin{pmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{pmatrix} \times \begin{pmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1.2 + (-2).4 + 3.2 & 1.3 + (-2).5 + 3.1 \\ -4.2 + 2.4 + 5.2 & -4.3 + 2.5 + 5.1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -4 \\ 10 & 3 \end{pmatrix}$$

and $BA = \begin{pmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{pmatrix}$

$$= \begin{pmatrix} 2.1 + 3.(-4) & 2.(-2) + 3.2 & 2.3 + 3.5 \\ 4.1 + 5.(-4) & 4.(-2) + 5.2 & 4.3 + 5.5 \\ 2.1 + 1.(-4) & 2.(-2) + 1.2 & 2.3 + 1.5 \end{pmatrix}$$

$$= \begin{pmatrix} -10 & 2 & 21 \\ -16 & 2 & 37 \\ -2 & -2 & 11 \end{pmatrix}$$

Hence, $AB \neq BA$.

Example 8:

Evaluate $A^2 - 4A - 5I$, where $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$ and

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Solution:

$$A^2 = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{pmatrix}$$

$$A^2 - 4A - 5I = \begin{pmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{pmatrix} - 4\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} - 5\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 9-4+5 & 8-8+0 & 8-8+0 \\ 8-8+0 & 9-4-5 & 8-8+0 \\ 8-8+0 & 8-8+0 & 9-4-5 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \phi, \text{ where } \phi \text{ is a null matrix.}$$

Questions for Review:

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. If $A = \begin{pmatrix} 1 & 2 \\ -3 & 0 \end{pmatrix}$, find $A^2 + 3A + 5I$ where I is unit matrix of order 2.

2. If $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 5 \\ 6 & 7 \end{pmatrix}$. Find a matrix C such that $A + B = 2C$.

3. If $A = \begin{pmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{pmatrix}$, find A^3.

4. Given $A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \\ 5 & 8 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 5 & 9 \\ 6 & -2 & 1 \end{pmatrix}$

   (i) Write down the order of the matrices *A* and *B*.
   (ii) Write down the order of the product *AB*.
   (iii) Calculate *AB*.
   (iv) Is it possible to calculate BA?
   (v) *Is AB = BA*?
   (vi) Are the following possible for operation?
   $A + B$, $A - B$, $2B$ and A^2

Lesson-3: Determinant

After studying this lesson, you should be able to:

- State the concept of determinant;
- Describe the advantages of determinant;
- Express the Cramer's rule;
- Solve the system of linear equations by Cramer's Rule.

Introduction

The present lesson is devoted to a brief discussion of determinants and their more elementary properties. The determinant concept is of a particular interest in solving simultaneous equations.

Determinant

An important concept in matrix algebra is that of the determinant. If a matrix is square, the elements of the matrix may be combined to compute a real-valued number called the determinant and is denoted either by the symbol Δ, or by placing vertical lines around the elements of the matrix (like **|A|**) or simply by *det.A*. The signs of the successive terms in the expansion of determinants will be alternately positive and negative until the last term is reached.

If a matrix is square, the elements of the matrix may be combined to compute a real-valued number called the determinant.

$$\text{If, } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Determinant of A will be denoted by $\Delta = |A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

Types of Determinants

First Order Determinant: A determinant of the first order is defined by the determinant of a 1×1 Matrix. The determinant of a 1×1 matrix is simply the value of the one element contained in the matrix.

Let, $A = [a_{11}]$ be a square matrix. Then $|A| = a_{11}$ be a determinant of first order.

Second Order Determinant: A determinant of the second order is defined by the determinant of a 2×2 Matrix.

Let, $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is a 2×2 matrix and the determinant of A is

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

That is the value of the determinant is given by the difference of the cross products.

Third Order Determinant: A determinant of the third order is defined by the determinant of a 3×3 Matrix.

Let, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is a 3×3 matrix and the determinant of A is

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Minors and Co-factors: The method discussed earlier applies for calculating the determinant of a 2×2 or 3×3 matrix. It does not, however, apply to matrices of higher dimensions. It is required a procedure for calculating a determinant that applies to any square matrix. This procedure is termed as the method of co-factor expansion. Before discussing the method of co-factor expansion, we must define two terms minor and co-factor.

Minors

The minor of an element is defined as a determinant by omitting the row and the column containing the element.

The minor of an element is defined as a determinant by omitting the row and the column containing the element. Thus, a minor is the determinant of the sub matrix formed by deleting the *i*-th row and *j*-th column of the matrix.

If a matrix, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

then – minor of $a_{11} = M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$

minor of $a_{12} = M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$

minor of $a_{13} = M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$ and so on.

Co-factors

A co-factor is a minor with a prescribed sign.

The co-factor of an element is the co-efficient of the element in the expanded form and is equal to the corresponding minor with proper sign. Thus, a co-factor is a minor with a prescribed sign. The rules for the sign of a co-factor of any element = $(-1)^{i+j} \times$ its minor, where i = number of row and j = number of column.

The co-factor of $a_{ij} = c_{ij} = (-1)^{i+j} M_{ij}$

For example, co-factor of $a_{11} = (-1)^{1+1} M_{11} = M_{11}$

co-factor of $a_{12} = (-1)^{1+2} M_{12} = -M_{12}$

Example-1:
Find the minors and co-factors of the elements at the 1st row of the determinant

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 0 \\ 3 & 2 & 7 \end{vmatrix}$$

Solution:

The minor of the element 1, i.e., a_{11} is $M_{11} = \begin{vmatrix} 5 & 0 \\ 2 & 7 \end{vmatrix} = 35$

The minor of the element 2, i.e., a_{12} is $M_{12} = \begin{vmatrix} 4 & 0 \\ 3 & 7 \end{vmatrix} = 28$

The minor of the element 3, i.e., a_{13} is $M_{13} = \begin{vmatrix} 4 & 5 \\ 3 & 2 \end{vmatrix} = -7$

The co-factor of 1, i.e., a_{11} is $C_{11} = (-1)^{1+1}\ .35 = 35$

The co-factor of 2, i.e., a_{12} is $C_{12} = (-1)^{1+2}\ .28 = 28$

The co-factor of 3, i.e., a_{13} is $C_{13} = (-1)^{1+3}\ (-7) = -7$

Expansion of Determinant and Use of Sarrus Diagram

Let $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

If the co-factor of a_{11}, a_{12} and a_{13} are A_{11}, A_{12} and A_{13} respectively, then

$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$

Sarrus Diagram: We can find out determinant value of a given matrix very conveniently by using Sarrus diagram. It is found by the following process:

We can find out determinant value of a given matrix by using Sarrus diagram.

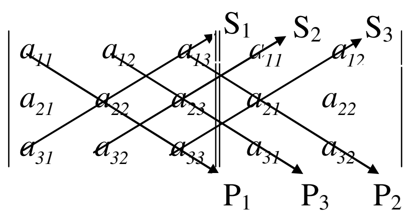
(i) Rewrite the first two columns of the matrix to the right of the original matrix.

(ii) Locate the elements on the three primary diagonals (P_1, P_2, P_3) and those on the three secondary diagonals (S_1, S_2, S_3).

(iii) Multiply the elements on each primary and each secondary diagonal.

(iv) The determinant equals the sum of the products for the three primary diagonals minus the sum of the products for the three secondary diagonals.

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, the determinant may be found by the following process



Thus, algebraically the determinant value is computed as

$$|A| = (a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}) - (a_{31}a_{22}a_{13} + a_{32}a_{23}a_{11} + a_{33}a_{21}a_{12})$$

Hence expansion of the determinant of $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ will be

$$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

$$= a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

Example-2:

Find the value of $\begin{vmatrix} 1 & 5 & 3 \\ 2 & 0 & 5 \\ -4 & 1 & -2 \end{vmatrix}$

Solution:

Let D = $\begin{vmatrix} 1 & 5 & 3 \\ 2 & 0 & 5 \\ -4 & 1 & -2 \end{vmatrix}$

= 1 (0 – 5) – 5 (– 4 + 20) + 3 (2 – 0)

= (–5 – 80 + 6) = 79.

Properties of Determinants

Certain properties hold for determinants. The following properties can be useful in computing the value of the determinant.

- If two rows or columns are interchanged in a determinant, the sign of the determinant changes but its value is unchanged.
- If rows are changed into columns and columns into rows, the determinant remains unchanged.
- If two rows or columns are identical in a determinant, it vanishes.

- If all the elements of any row or column are zero, the determinant is zero.
- If any multiple of one row or column is added to another row or column, the value of the determinant is unchanged.
- If any row or column is a multiple of another row or column, the determinant equals to zero.

Example-3:

Show that $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

Solution:

Applying $C'_1 = C_1 - C_2$; $C'_2 = C_2 - C_3$ we get

$$\begin{vmatrix} 0 & 0 & 1 \\ a - b & b - c & c \\ a^2 - b^2 & b^2 - c^2 & c^2 \end{vmatrix}$$

$$= (a-b)(b-c)\begin{vmatrix} 1 & 1 \\ a+b & b+c \end{vmatrix}$$

$$= (a-b)(b-c)(c-a)$$

Example-4:

Show that

$$\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3$$

Solution:

Applying $C'_1 = C_1 + C_2 + C_3$, we get

$$= \begin{vmatrix} 2a+2b+2c & a & b \\ 2a+2b+2c & b+c+2a & b \\ 2a+2b+2c & a & c+a+2b \end{vmatrix}$$

$$= 2(a+b+c)\begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & c+a+2b \end{vmatrix}$$

Applying $R'_1 = R_1 - R_2$; $R'_2 = R_2 - R_{23}$

$$= 2(a+b+c)\begin{vmatrix} 0 & -(a+b+c) & 0 \\ 0 & (a+b+c) & -(a+b+c) \\ 1 & a & c+a+2b \end{vmatrix}$$

$$= 2(a+b+c)^3$$

Example-5:

Show that $\begin{vmatrix} 1 & x & y+z \\ 1 & y & z+x \\ 1 & z & x+y \end{vmatrix} = 0$

Solution:

Applying $C_3' = C_2 + C_3$ we get,

$$= \begin{vmatrix} 1 & x & x+y+z \\ 1 & y & x+y+z \\ 1 & z & x+y+z \end{vmatrix}$$

$$= (x+y+z)\begin{vmatrix} 1 & x & 1 \\ 1 & y & 1 \\ 1 & z & 1 \end{vmatrix}$$

$= 0.$

Example-6:

Solve the equation $\begin{vmatrix} 1 & 1 & 1 \\ x & a & b \\ x^3 & a^3 & b^3 \end{vmatrix} = 0$

Solution:

Applying $C'_2 = C_2 - C_1$; $C'_3 = C_3 - C_2$; we get

$$\begin{vmatrix} 1 & 0 & 0 \\ x & a - x & b - a \\ x^3 & a^3 - x^3 & b^3 - a^3 \end{vmatrix} = 0$$

$$\begin{vmatrix} a - x & b - a \\ a^3 - x^3 & b^3 - a^3 \end{vmatrix} = 0$$

or, $(a-x)(b-a)(b^2 + ab + a^2 - a^2 - ax - x^2) = 0$

or, $(a-x)(b-a)(b^2 + ab - ax - x^2) = 0$

or, $-(a-x)(b-a)(x^2 + ax - ab - b^2) = 0$

or, $(a-x)(b-a)(x^2 + ax - ab - b^2) = 0$

$\therefore\ x = a \ \ or \ \ x = \dfrac{-a \pm \sqrt{a^2 - 4(-ab - b^2\ )}}{2}$

or, $x = a \ \ or \ \ x = \dfrac{-a \pm \sqrt{a^2 + 4ab + b^2\ )}}{2}$

$\therefore\ x = a, b, -(a+b)$

Cramer's Rule and Its use in the Solution of Equations

Cramer's rule is a simple rule using determinants to express the solution of a system of linear equations for which the number of equations is equal to the number of variables. This rule states $\bar{x}_i = \frac{D_i}{D}$ where x_i is the *i*-th unknown variable in a series of equations, D is the determinant of the coefficient matrix, and D_i is the determinant of a special matrix formed from the original coefficient matrix by replacing the column of coefficients of x_i with the column vector of constants. Thus, Cramer's rule can be fruitfully applied in case $D \neq 0$.

Example-7:

Solve the following system of equations by using Cramer's Rule.

$$5x - 6y + 4z = 15$$
$$7x + 4y - 3z = 19$$
$$2x + y + 6z = 46$$

Solution:

Here $D = \begin{vmatrix} 5 & -6 & 4 \\ 7 & 4 & -3 \\ 2 & 1 & 6 \end{vmatrix} = 419$

$$D_x = \begin{vmatrix} 15 & -6 & 4 \\ 19 & 4 & -3 \\ 46 & 1 & 6 \end{vmatrix} = 1257$$

$$D_y = \begin{vmatrix} 5 & 15 & 4 \\ 7 & 19 & -3 \\ 2 & 46 & 6 \end{vmatrix} = 1676$$

$$D_z = \begin{vmatrix} 5 & -6 & 15 \\ 7 & 4 & 19 \\ 2 & 1 & 46 \end{vmatrix} = 2514$$

We know from the Cramer's Rule, $\frac{x}{D_x} = \frac{y}{D_y} = \frac{z}{D_z} = \frac{1}{D}$

Hence $x = \frac{D_x}{D} = \frac{1257}{419} = 3$

$$y = \frac{D_y}{D} = \frac{1676}{419} = 4$$

$$z = \frac{D_z}{D} = \frac{2514}{419} = 6.$$

Example-8:

Solve the following system of equations by using Cramer's Rule.

$$x + y + z = 9$$
$$2x + 5y + 7z = 52$$
$$2x + y - z = 0$$

Solution:

$$\text{Here } D = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{vmatrix} = -4$$

$$D_x = \begin{vmatrix} 9 & 1 & 1 \\ 52 & 5 & 7 \\ 0 & 1 & -1 \end{vmatrix} = -4$$

$$D_y = \begin{vmatrix} 1 & 9 & 1 \\ 2 & 52 & 7 \\ 2 & 0 & -1 \end{vmatrix} = -12$$

$$D_z = \begin{vmatrix} 1 & 1 & 9 \\ 2 & 5 & 52 \\ 2 & 1 & 0 \end{vmatrix} = -20$$

We know from the Cramer's Rule, $\frac{x}{D_x} = \frac{y}{D_y} = \frac{z}{D_z} = \frac{1}{D}$

Hence $x = \frac{D_x}{D} = \frac{-4}{-4} = 1$

$$y = \frac{D_y}{D} = \frac{-12}{-4} = 3$$

$$z = \frac{D_z}{D} = \frac{-20}{-4} = 5.$$

Questions for Review:

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. Find all the minors and co-factors of the following determinant

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 7 \\ -2 & 8 & 1 \end{vmatrix}$$

2. Show that $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

3. Show that $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

4. Find the value of $\begin{vmatrix} x+y & x & y \\ x & x+z & z \\ y & z & y+z \end{vmatrix}$

5. Solve the following system of equations by using Cramer's Rule:

$$x+5y-z=9$$
$$3x-3y+2z=7$$
$$2x-4y+3z=1$$

6. Solve the following system of equations by using Cramer's Rule:

$$x-y+z=1$$
$$x+y-2z=0$$
$$2x-y-z=0$$

7. Solve the equation $\begin{vmatrix} p+x & q+x & r+x \\ q+x & r+x & p+x \\ r+x & p+x & q+x \end{vmatrix} = 0$

Lesson-4: Matrix Inversion

After studying this lesson, you should be able to:

- Explain inverse matrix;
- Solve system of linear equations by inverse matrix method.

Introduction

The operation of dividing one matrix directly by another does not exist in matrix theory but equivalent of division of a unit matrix by any square matrix can be accomplished (in most cases) by a process known as inversion of matrix. The concept of inverse matrix is useful in solving simultaneous equations, input-output analysis and regression analysis.

If A is a square matrix of order n, then a square matrix B of the same order n is said to be inverse of A if AB = BA = I (unit matrix).

Inverse Matrix

If A is a square matrix of order n, then a square matrix B of the same order n is said to be inverse of A if $AB = BA = I$ (unit matrix).

Methods of Matrix Inversion

There are several methods for determining the inverse of a matrix; two of these are discussed in below.

(i) Co-factor matrix method.
(ii) Gauss- Jordan Elimination method.

Working Rule for Inverse Matrix (Co-factor matrix method)

To evaluate the inverse of a square matrix A, we should follow the following steps:

(i) Evaluate |A| for the matrix A and be sure that $|A| \neq 0$

(ii) Calculate the co-factors of all the elements of the matrix A.

(iii) Find the matrix of the co-factor A^C.

(iv) Then find the Adjoint of A by taking transpose of A^C such that Adj A = $(A^C)^T$.

(v) Finally divide all the elements of Adj A by |A| to get the required inverse A^{-1}.

Example-1:

Find the inverse of the matrix, $A = \begin{bmatrix} 2 & 4 \\ 3 & 8 \end{bmatrix}$

Solution:

The determinant of the matrix A is, $|A| = \begin{vmatrix} 2 & 4 \\ 3 & 8 \end{vmatrix} = 4 \neq 0$

The co-factor matrix of A is, $A^C = \begin{bmatrix} 8 & -3 \\ -4 & 2 \end{bmatrix}$

The Ad joint matrix of A is, $A^j = \begin{bmatrix} 8 & -4 \\ -3 & 2 \end{bmatrix}$

Therefore, the inverse of A is,

$$A^{-1} = \frac{1}{\Delta} A^{j} = \frac{1}{4}\begin{bmatrix} 8 & -4 \\ -3 & 2 \end{bmatrix}$$

Example-2:

Find the inverse of the matrix, A = $\begin{bmatrix} 1 & 2 & 0 \\ 1 & 0 & -1 \\ -1 & 3 & 2 \end{bmatrix}$

Solution:

The determinant of the matrix A is, $|A| = \begin{vmatrix} 1 & 2 & 0 \\ 1 & 0 & -1 \\ -1 & 3 & 2 \end{vmatrix} = 1$

The co-factor matrix of A is, $A^{C} = \begin{bmatrix} 3 & -1 & 3 \\ -4 & 2 & -5 \\ -2 & 1 & -2 \end{bmatrix}$

The Adjoint matrix of A is, $A^{J} = \begin{bmatrix} 3 & -4 & -2 \\ -1 & 2 & 1 \\ 3 & -5 & -2 \end{bmatrix}$

Therefore, the inverse of A is,

$$A^{-1} = \frac{1}{\Delta} A^{J} = \frac{1}{1}\begin{bmatrix} 3 & -4 & -2 \\ -1 & 2 & 1 \\ 3 & -5 & -2 \end{bmatrix} = \begin{bmatrix} 3 & -4 & -2 \\ -1 & 2 & 1 \\ 3 & -5 & -2 \end{bmatrix}$$

Gauss-Jordan Elimination Method

To determine the inverse of an $m \times m$ matrix 'A', following are the steps

(i) Determining the determinant value of A, whether it is non-singular or not.

(ii) Augmenting the matrix A with an $m \times m$ identity matrix, resulting in ($A \,|\, I$).

(iii) Performing row operations on the entire augmented matrix so as to transform 'A' into an $m \times m$ identify matrix. The resulting matrix will have the following form ($I \,|\, A^{-1}$) where, the A^{-1} can be read to the right of the vertical line.

Example-3:

Find the inverse of the matrix, $A = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}$

Solution:

Augmented the matrix 'A' by 2×2 identity matrix, we get –

$$\begin{bmatrix} 3 & 7 & | & 1 & 0 \\ 2 & 5 & | & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{7}{3} & | & \frac{1}{3} & 0 \\ 2 & 5 & | & 0 & 1 \end{bmatrix} \text{applying } r_1^{/} = r_1 \times \frac{1}{3}$$

$$\begin{bmatrix} 1 & \frac{7}{3} & | & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & | & -\frac{2}{3} & 1 \end{bmatrix} \text{applying, } r_2^{/} = r_2 - r_1 \times 2$$

$$\begin{bmatrix} 1 & \frac{7}{3} & | & \frac{1}{3} & 0 \\ 0 & 1 & | & -2 & 3 \end{bmatrix} \text{applying, } r_2^{/} = r_2 \times \frac{1}{3}$$

$$\begin{bmatrix} 1 & 0 & | & 5 & -7 \\ 0 & 1 & | & -2 & 3 \end{bmatrix} \text{ applying, } r_1^{/} = r_1 - r_2 \times \frac{7}{3}$$

So, the inverse of 'A' is, $A^{-1} = \begin{bmatrix} 5 & -7 \\ -2 & 3 \end{bmatrix}$

Solution of Linear Equations by Using Inverse of Matrix

Matrix algebra permits the concise expression of a system of linear equations.

Matrix algebra permits the concise expression of a system of linear equations. The inverse matrix can be used to solve a system of simultaneous equations. Let a system of linear equations are:

$$a_{11}x + a_{12}y + a_{13}z = k_1$$
$$a_{21}x + a_{22}y + a_{23}z = k_2$$
$$a_{31}x + a_{32}y + a_{33}z = k_3$$

It can be written in the matrix form as follows:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}$$

$AX = B$; where, A = $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, B = $\begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}$

$$X = A^{-1}B$$

Example-4:

Use matrix inversion to solve the following system of equations

$$4x_1 + x_2 - 5x_3 = 8$$
$$-2x_1 + 3x_2 + x_3 = 12$$
$$3x_1 - x_2 + 4x_3 = 5$$

Solution:

The given system of equations can be written in the matrix form

$$\begin{bmatrix} 4 & 1 & -5 \\ -2 & 3 & 1 \\ 3 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 12 \\ 5 \end{bmatrix}$$

$$X = A^{-1}B$$

$$\text{Now } |A| = \begin{vmatrix} 4 & 1 & -5 \\ -2 & 3 & 1 \\ 3 & -1 & 4 \end{vmatrix} = 98$$

The co-factor matrix of A is $A^C = \begin{bmatrix} 13 & 11 & -7 \\ 1 & 31 & 7 \\ 16 & 6 & 14 \end{bmatrix}$

The Adjoint matrix of A is, $A_j = \begin{bmatrix} 13 & 1 & 16 \\ 11 & 31 & 6 \\ -7 & 7 & 14 \end{bmatrix}$

$\therefore$ The inverse of A is, $A^{-1} = \frac{1}{\Delta} A_j = \frac{1}{98} \begin{bmatrix} 13 & 1 & 16 \\ 11 & 31 & 6 \\ -7 & 7 & 14 \end{bmatrix}$

$$X = A^{-1}B = \frac{1}{98} \begin{bmatrix} 13 & 1 & 16 \\ 11 & 31 & 6 \\ -7 & 7 & 14 \end{bmatrix} \begin{bmatrix} 8 \\ 12 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$$

$\therefore$ $x_1 = 2, x_2 = 5, x_3 = 1.$

Example-5:

Solve the following system of equations by using Gaussian method.

$$x + y + z = 7$$
$$x + 2y + 3z = 16$$
$$x + 3y + 4z = 22$$

Solution:

Given system of equations in matrix form

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 16 \\ 22 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 9 \\ 15 \end{bmatrix}$$; Applying $R_2 = R_2 - R_1$; $R_3 = R_3 - R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 9 \\ -3 \end{bmatrix}$$; Applying $R_3 = R_3 - 2R_2$

$$x + y + z = 7$$

Hence $$y + 2z = 9$$

$$-z = -3$$

Thus, $z = 3$, $y = 3$, $x = 1$.

Example-6:

Solve the following system of equations by using Gaussian method.

$$2x - 5y + 7z = 6$$
$$x - 3y + 4z = 3$$
$$3x - 8y + 11z = 11$$

Solution:

Given system of equations in matrix form

$$\begin{bmatrix} 2 & -5 & 7 \\ 1 & -3 & 4 \\ 3 & -8 & 11 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 11 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & 4 \\ 2 & -5 & 7 \\ 3 & -8 & 11 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 11 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & 4 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$$; Applying $R_2 = R_2 - 2R_1$; $R_3 = R_3 - 3R_1$

$$\begin{bmatrix} 1 & -3 & 4 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$$; Applying $R_3 = R_3 - R_2$

$$x - 3y + 4z = 3$$

Hence $$y - z = 0$$

$$0 = 2$$

Since, 0 = 2 is false, the given system of equations has no solution. So given system of equations is inconsistent.

Questions for Review:

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. Find the inverse of the matrix, $A = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 1 \\ 5 & 2 & 2 \end{bmatrix}$

2. Find the inverse of the matrix, $A = \begin{bmatrix} 1 & 3 & 4 \\ 3 & -1 & 6 \\ -1 & 5 & 1 \end{bmatrix}$

3. Solve the following system of equations by using Gaussian method.

$$2x - 5y + 7z = 6$$
$$x - 3y + 4z = 3$$
$$3x - 8y + 11z = 11$$

4. Use matrix inversion to solve the following system of equations:

$$x + y + z = 3$$
$$x + 2y + 3z = 4$$
$$x + 4y + 9z = 6$$

5. Use matrix inversion to solve the following system of equations:

$$x + 2y + 3z = 6$$
$$2x + 4y + z = 7$$
$$3x + 2y + 9z = 14$$

Lesson-5: Application of Matrices in Business

After studying this lesson, you should be able to:

- Develop matrices by using given business information;
- Apply the concepts of matrices to solve the business problems.

Introduction

Matrix algebra permits the concise expression of a system of linear equations.

Matrix is the powerful tool in modern mathematics having wide applications. Demographers, sociologists, economists use matrices in different way. Many economic relationships can be approximated by linear equations. Matrix algebra permits the concise expression of a system of linear equations. Let us know few applications of matrices in business.

Illustrative Example

Example-1:

A manufacturer produces three products A, B, C that he sells in the market. Annual sales volumes are indicated as follows:

Market	Products		
	A	B	C
I	8000	10000	15000
II	10000	2000	20000

(i) If unit sale prices of A, B and C are \$2.25, \$1.50 and 1.25 respectively, find the total revenue in each market with the help of matrices.

(ii) If the unit costs of the above three products are \$1.60, \$1.20 and \$0.90 respectively, find the gross profit with the help of matrices.

Solution:

(i) The total revenue in each market is given by the product matrix:

$$\begin{pmatrix} 2.25 & 1.50 & 1.25 \end{pmatrix} \begin{pmatrix} 8000 & 10000 \\ 10000 & 2000 \\ 15000 & 20000 \end{pmatrix}$$

$$= [51750 \quad 50500]$$

The total revenue from the market I is \$51750 and the total revenue from the market II is \$50500.

(ii) The total cost of products with the manufacturer sells in the markets are:

$$\begin{pmatrix} 1.60 & 1.20 & 0.90 \end{pmatrix} \begin{pmatrix} 8000 & 10000 \\ 10000 & 2000 \\ 15000 & 20000 \end{pmatrix}$$

$$= [38300 \quad 36400]$$

The total cost of products that the manufacturer sells in the market I and II are \$38300 and \$36400 respectively.

Required gross profit = (Total revenue received from both the markets) – (Total cost of products that the manufacturer sold in both the market)

= (51750 + 50500) – (38300 + 36400)

= 102250 – 74700

= 27550.

Example-2:

A finance company has offices located in every division, every district and every thana. Assume that there are five divisions, thirty districts and two hundred thanas. Each office has one head clerk, one cashier, one clerk and one peon. A divisional office has in addition one office superintendent, two clerks, one typist and one peon. A district office has in addition one clerk and one peon. The basic monthly salaries are as follows: office superintendent \$500, head clerk \$200, cashier \$175 clerks and typists \$150 and peon \$100. Using matrix notation, find the following

(i) The total number of posts of each kind in all the offices taken together.

(ii) The total basic monthly salary bill of each kind of office and

(iii) The total basic monthly salary bill of all the offices taken together.

Solution:

Let the number of offices can be arranged as elements of a row matrix

$$A = \begin{pmatrix} 5 & 30 & 200 \end{pmatrix}$$

The composition of staff in various offices can be arranged in a 3×6 matrix

$$B = \begin{pmatrix} 1 & 1 & 1 & 3 & 1 & 2 \\ 0 & 1 & 1 & 2 & 0 & 2 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}$$

The basic monthly salaries of various types of employees of these offices correspond to the elements of the column matrix, $C = \begin{pmatrix} 500 \\ 200 \\ 175 \\ 150 \\ 150 \\ 100 \end{pmatrix}$

(i) Total numbers of posts of each kind in all the offices are the elements of the product matrix AB.

$$AB = \begin{pmatrix} 5 & 30 & 200 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 3 & 1 & 2 \\ 0 & 1 & 1 & 2 & 0 & 2 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}$$

$$= [5 \quad 235 \quad 235 \quad 275 \quad 5 \quad 270]$$

Thus, the required numbers of posts in all the offices taken together are 5-office superintendent, 235 head clerks, 235 cashiers, 275 clerks, 5 typists and 270 peons.

(ii) $$\begin{pmatrix} 1 & 1 & 1 & 3 & 1 & 2 \\ 0 & 1 & 1 & 2 & 0 & 2 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 500 \\ 200 \\ 175 \\ 150 \\ 150 \\ 100 \end{pmatrix} = \begin{pmatrix} 1675 \\ 875 \\ 625 \end{pmatrix}$$

Thus, the total basic monthly salary bill of each divisional, district and Thana offices are \$1675, \$875 and \$625 respectively.

(iii) Total basic monthly salary bill of all the offices is the element of the product matrix ABC,

$$\text{i.e., } ABC = \begin{pmatrix} 5 & 30 & 200 \end{pmatrix} \times \begin{pmatrix} 1675 \\ 875 \\ 625 \end{pmatrix} = 159625.$$

Thus, the total basic monthly salary bill of all the offices taken together is \$159625.

Example-3:

Three persons *A*, *B* and *C* posses Tk.3000, Tk.2000 and Tk.2500 respectively. *A* with his entire amount purchased 5 shares of Tk.*X* each, 3 shares of Tk.*Y* each and 4 shares of Tk.*Z* each. *B* purchased 3 shares of Tk.*X* each, 4 shares of Tk.*Y* each and 2 shares of Tk.*Z* each with his entire amount and *C* purchased 4 shares of Tk.*X* each, 3 shares of Tk.*Y* each and 4 shares of Tk.*Z* each with his entire amount. Determine the value of each share of different types.

Solution:

We have, $5x + 3y + 4z = 3000$

$$3x + 4y + 2z = 2000$$

$$4x + 3y + 4z = 2500$$

$$D = \begin{vmatrix} 5 & 3 & 4 \\ 3 & 4 & 2 \\ 4 & 3 & 4 \end{vmatrix} = 10$$

$$D_x = \begin{vmatrix} 3000 & 3 & 4 \\ 2000 & 4 & 2 \\ 2500 & 3 & 4 \end{vmatrix} = 5000$$

$$D_y = \begin{vmatrix} 5 & 3000 & 4 \\ 3 & 2000 & 2 \\ 4 & 2500 & 4 \end{vmatrix} = 1000$$

$$D_z = \begin{vmatrix} 5 & 3 & 3000 \\ 3 & 4 & 2000 \\ 4 & 3 & 2500 \end{vmatrix} = 500$$

From Cramer's rule we know that,

We know from the Cramer's Rule, $\dfrac{x}{D_x} = \dfrac{y}{D_y} = \dfrac{z}{D_z} = \dfrac{1}{D}$

Hence $x = \dfrac{D_x}{D} = \dfrac{5000}{10} = 500$

$y = \dfrac{D_y}{D} = \dfrac{1000}{10} = 100$

$z = \dfrac{D_z}{D} = \dfrac{500}{10} = 50.$

Example-4:

To control a certain crop disease it is necessary to use 7 units of chemical *A*, 10 units of chemical *B* and 6 units of chemical *C*. One barrel of spray *P* contains 1 unit of *A*, 4 units of *B* and 2 units of *C*. One barrel of spray *Q* contains 3 units of *A*, 2 units of *B*, and 2 units of *C*. One barrel of spray *R* contains 4 units of *A*, 3 units of *B* and 2 units of *C*. How much of each type of spray should be used to control the disease?

Solution:

Let *x* barrels of spray *P*, *y* barrels of spray *Q* and *z* barrels of spray *R* be used to control the disease. Then we can write,

$$x + 3y + 4z = 7$$
$$4x + 2y + 3z = 10$$
$$2x + 2y + 2z = 6$$

The given information can be written under the matrix form as follows:

$$\begin{bmatrix} 1 & 3 & 4 \\ 4 & 2 & 3 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 10 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 3 & 4 \\ 4 & 2 & 3 \\ 2 & 2 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 7 \\ 10 \\ 6 \end{bmatrix}$$

$$\text{Let } A = \begin{bmatrix} 1 & 3 & 4 \\ 4 & 2 & 3 \\ 2 & 2 & 2 \end{bmatrix}$$

The determinant of the matrix *A* is, $|A| = \begin{vmatrix} 1 & 3 & 4 \\ 4 & 2 & 3 \\ 2 & 2 & 2 \end{vmatrix} = 8$

The co-factor matrix of *A* is, $A^C = \begin{bmatrix} -2 & -2 & 4 \\ 2 & -6 & 4 \\ 1 & 13 & -10 \end{bmatrix}$

The Adjoint matrix of A is, $A^J = \begin{bmatrix} -2 & 2 & 1 \\ -2 & -6 & 13 \\ 4 & 4 & -10 \end{bmatrix}$

Therefore, the inverse of *A* is,

$$A^{-1} = \frac{1}{\Delta} A^J = \frac{1}{8} \begin{bmatrix} -2 & 2 & 1 \\ -2 & -6 & 13 \\ 4 & 4 & -10 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{8} \begin{bmatrix} -2 & 2 & 1 \\ -2 & -6 & 13 \\ 4 & 4 & -10 \end{bmatrix} \begin{bmatrix} 7 \\ 10 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{8} \begin{bmatrix} 12 \\ 4 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3/2 \\ 1/2 \\ 1 \end{bmatrix}$$

Hence $1\frac{1}{2}$ barrels of the spray P, $\frac{1}{2}$ barrel of spray Q and 1 barrel of spray R should be used to control the disease.

Example-5:

The cost of manufacturing the three types of motorcars is given below:

Car	Labor hours	Material used	Subcontracted works
A	40	100	50
B	80	150	80
C	100	250	100

Labor cost \$2 per hour, per unit material cost is \$0.5 and one unit of subcontracted work costs \$1. Find the total cost of manufacturing 3000, 2000 and 1000 vehicles of type *A*, *B*, *C* respectively. If the selling prices of car *A*, *B*, *C* are \$2000, 3500 and \$4500 respectively, then find the profit from selling those cars.

Solution:

Consider the following matrices,

$$M = \begin{bmatrix} 40 & 100 & 50 \\ 80 & 150 & 80 \\ 100 & 250 & 100 \end{bmatrix} \qquad N = \begin{bmatrix} 2 \\ 10 \\ 1 \end{bmatrix}$$

$$MN = \begin{bmatrix} 180 \\ 315 \\ 425 \end{bmatrix}$$

This column matrix represents cost of each car A, B, C in that order.

Let $P = (3000 \quad 2000 \quad 1000)$, this row matrix represents number of cars A, B, C to be manufactured in that order.

Now $PMN = (1595000)$

Thus total cost of manufacturing three cars A, B, C is \$1595000.

Let $Q = \begin{bmatrix} 2000 \\ 3500 \\ 4500 \end{bmatrix}$; this column matrix represents the selling price of A, B, C.

Now, Total Revenue = PQ = $(3000 \quad 2000 \quad 1000) \begin{bmatrix} 2000 \\ 3500 \\ 4500 \end{bmatrix}$

$= 17500000$

Profit = Total revenue – Total cost = 17500000 – 1595000 = 15905000.

Questions for Review

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. *A*, *B* and *C* has Tk.480, Tk.760 and Tk.710 respectively. They utilized the amounts to purchase three types of shares of prices x, y and z respectively. *A* purchases 2 share of price x, 5 of price y and 3 of price z. *B* purchases 4 shares of price x, 3 of price y and 6 of price z. *C* purchases 1 share of price x, 4 of price y and 10 of price z. Find the value of x, y and z.

2. A manufacturing unit produces three types of television sets *A*, *B*, *C*. The following matrix shows the sale of television sets in two different cities.

$$\begin{pmatrix} A & B & C \\ 400 & 300 & 200 \\ 300 & 200 & 100 \end{pmatrix}$$

   If cost price of each set *A*, *B*, *C* is Tk.1000, Tk.2000, and Tk.3000 respectively and selling prices are Tk.1500, Tk.3000, Tk.4000 respectively, find the total profit using matrix algebra only.

3. The following matrix represents the results of the examination of MBA.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix}$$

   The rows represent the three sections of the class. The first three columns represent the number of students securing 1^{st}, 2^{nd} and 3^{rd} divisions respectively in that order and fourth column represents the number of students who failed in the examination.

   (i) How many students passed in three sections respectively?
   (ii) How many students failed in three sections respectively?
   (iii) Write down the matrix in which number of successful students is shown.
   (iv) Write down the column matrix where only failed students are shown.
   (v) Write down the column matrix showing students in the 1^{st} division from three sections.

4. A publishing house has two branches. In each branch, there are three offices. In each office, there are 3 peons, 4 clerks and 5 typists. In one office of a branch, 6 salesmen are also working. In each office of other branch 2 head clerks are also working. Using matrix notation find
   (i) the total number of posts of each kind in all the offices taken together in each branch.
   (ii) the total number of posts of each kind in all the offices taken together from both the branches.

Applications to Economics and Business

12

This unit is designed to introduce the learners to the basic concepts associated with the applications of mathematics in business and economics. The learners will learn about different types of functions that are widely used in economics and the relationships among total, average and marginal functions. This unit also discusses the elasticity of demand and supply, consumers' surplus and producers' surplus etc. Some relevant examples are provided in this unit for clear understanding to the learners.

Blank Page

Lesson-1: Uses of Different Functions in Business and Economics

After studying this lesson, you should be able to:

- Develop different functions relating to demand, supply, cost, revenue, profit and production;
- Analyze different types of functions;
- Determine total cost, average cost and marginal cost;
- Determine total revenue and marginal revenue;
- Determine total profit and marginal profit.

Introduction

Functions explain the nature of correspondence between variables indicated by some formula, graph or a mathematical equation. A function is a term used to symbolize relationship between or among the variables. When two variables are so related that for any arbitrarily assigned value to one of them there corresponds a definite value or a set of definite values for the other, the second variable is said to be the function of the first. We shall now introduce some different types of functions, which are particularly useful in business and economics.

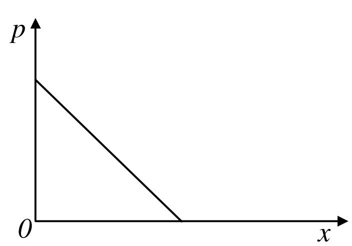
A function is a term used to symbolize relationship between or among the variables.

Demand Function

Demand functions are an essential concept in the study of economics. Usually these functions are curves rather than straight lines, but straight lines provide good illustrations of demand characteristics. The demand function specifies the amounts of a particular commodity that buyers are willing and able to purchase at each price in a series of possible prices during a specified period of time. Demand refers to the relationship between price and quantity. Conventionally, we plot price on the vertical axis and quantity demanded on the horizontal axis.

Demand refers to the relationship between price and quantity.

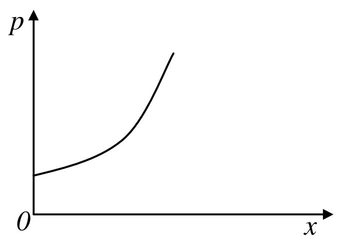
Let p be the price and x be the quantity demanded, the function $x = f(p)$ is plotted as a demand curve. It usually slopes downwards as demand decreases when price increases.



Supply Function

Supply function specifies the amounts of a particular product that a producer is willing and able to produce and make available for sale at each price in a series of possible prices during a specified period of time.

Let p be the price and x be the quantity supplied; the function $x = g(p)$ is plotted as a supply curve. When price increases quantity supplied increases, therefore, a supply curve slopes upward.



Total Revenue (TR)

A firm's total revenue that is derived from the sale of a product is given by price multiplied by quantity. It can be expressed mathematically, $TR = p \times q$.

From the demand function we observe that price is a function of quantity,

i.e., $p = f(q)$

Total revenue is thus represented as a function of quantity,

i.e., $TR = f(q) \times q$

Total Cost (TC)

The cost of production to the firm depends upon the costs of inputs used in the production process and the quantity of product manufactured.

The cost of production to the firm depends upon the costs of inputs used in the production process and the quantity of product manufactured. If x is the quantity produced of a certain product by a firm at a total cost C, we can write the total cost function: $TC = f(x)$.

It may be noted that the total cost TC of producing goods can be analyzed into two parts: (i) fixed cost which is independent of x with certain limits, and (ii) variable cost depending on x. Thus, we may have cost function of the type, $TC = FC + VC$, where FC is fixed cost and VC is variable cost.

Cost curves are obtained from the knowledge of production functions. Usually, the cost curve is rising to the right as the cost of production generally increases with the output (x).

Total Profit

Profits of a firm depend upon both revenue and cost.

Profits of a firm depend upon both revenue and cost. Profits are defined as the excess of total revenue over total costs. Symbolically, it can be expressed as, $P = TR - TC$.

Average Revenue (AR)

The average revenue from a product is found by dividing the total revenue by the quantity of the product sold. The function that describes average revenue is the quotient of the total revenue function and the quantity.

We see that average revenue function and the demand function are equivalent.

$$AR = \frac{TR}{q} = \frac{f(q).q}{q} = f(q)$$

Average Cost (AC)

Average cost of production or cost per unit is obtained by dividing total cost by the quantity produced.

$$AC = \frac{C}{x}$$

Marginal Revenue (MR)

Marginal revenue is the additional revenue derived from selling one more unit of a product or service. If each unit of a product sells at the same price, the marginal revenue is always equal to the price.

Marginal revenue is the additional revenue derived from selling one more unit of a product or service.

Thus, $MR = \dfrac{d(TR)}{dq}$; the rate of change of revenue with respect to units of output.

Marginal Cost (MC)

Marginal cost is defined as the change in total cost incurred in the production of an additional unit.

$MC = \dfrac{d(TC)}{dq}$; the rate of change of cost with respect to units of production.

Marginal Profit (MP)

Marginal profit analysis is concerned with the effect on profit if one additional unit of a product is produced and sold. As long as the additional revenue brought in by the next unit exceeds the cost of producing and selling that unit, there is a net profit from producing and selling that unit and total profit increases. If, however, the additional revenue from selling the next unit is exceeded by the cost of producing and selling the additional unit, there is a net loss from that next unit and total profit decreases.

Marginal profit analysis is concerned with the effect on profit if one additional unit of a product is produced and sold.

$MP = \dfrac{d(TP)}{dq}$; the rate of change of profit with respect to units of output.

A rule of thumb concerning whether or not to produce an additional unit (assuming profit maximization is of greatest importance) is given below.

(i) If $MR > MC$, produce the next unit.

(ii) If $MR < MC$, do not produce the next unit.

(iii) If $MR = MC$, the total profit will be maximized.

Production Function

A production function is a technical relationship between the inputs of production and the output of the firm's.

A production function is a technical relationship between the inputs of production and the output of the firm's. The relationship is such that the level of output depends upon the level of inputs used, not vice versa. A production function can be written as: $Q = f(L, K)$, where L and K are quantities of labor and capital respectively required to produce Q.

In Economics, the Cobb – Douglas production function defined as

$$Q = aL^{\alpha}K^{\beta}, \text{ where } \alpha + \beta = 1.$$

Utility Function

Utility is the power of a commodity to satisfy human want.

The term utility refers to the benefit or satisfaction or pleasure of a person gets from the consumption of a commodity or service. In abstract sense, utility is the power of a commodity to satisfy human want, i.e., utility is want-satisfying power. A commodity is likely to have utility if it can satisfy a want. For example, bread has the power to satisfy hunger, water quenches our thirst and so on.

If $U(x, y)$ denotes the satisfaction obtained by an individual when he buys quantities x and y of two commodities X and Y respectively, then $U(x, y)$ is called the utility function or utility index of the individual.

Consumption Function

If C is the total consumption of the community dependent on income Y and propensity to consume c, the aggregate consumption function is defined by

$$C = a + cY$$

But since $Y = C + S$

$$S = Y - (a + cY)$$; This is the savings function of the community.

Relationship between Average Cost (AC) and Marginal Cost (MC)

Let us assume that total cost, $C = f(x)$

Thus, $AC = \dfrac{C}{x}$

$$\frac{d}{dx}(AC) = \frac{d}{dx}\left(\frac{C}{x}\right) = \frac{x\frac{dC}{dx} - C}{x^2} = \frac{1}{x}\left(\frac{dC}{dx} - \frac{C}{x}\right) = \frac{1}{x}(MC - AC)$$

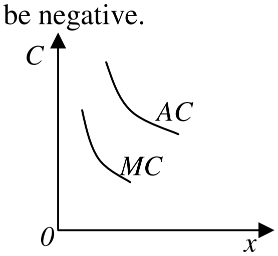
Case 1: When average cost curve slopes downwards, i.e., when AC is declining, its slope will be negative.

$$\frac{d}{dx}(AC) < 0$$

$$\frac{1}{x}(MC - AC) < 0$$

$$MC - AC < 0$$

MC < AC



Thus when AC curves slopes downwards MC curve will lie below AC curve.

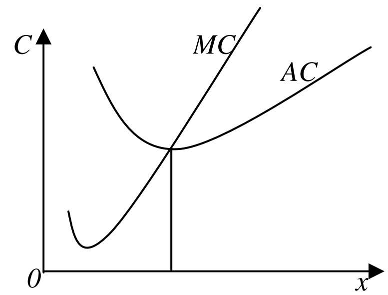
Case 2: When AC curve reaches a minimum point, its slope becomes zero.

$$\frac{d}{dx}(AC) = 0$$

$$\frac{1}{x}(MC - AC) = 0$$

MC – AC = 0

MC = AC



Thus MC curve and AC curve intersect at the point of minimum average cost.

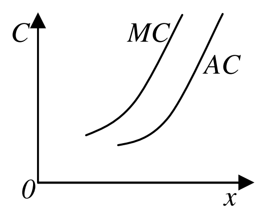
Case 3: When average cost curve slopes upward

$$\frac{d}{dx}(AC) > 0$$

$$\frac{1}{x}(MC - AC) > 0$$

$$MC - AC > 0$$

MC >AC



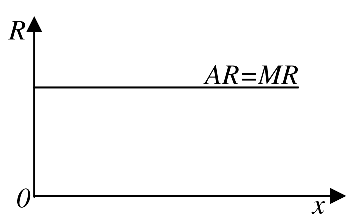
Thus when AC curve slopes upward MC curve will be above AC curve.

Relationship between Average Revenue (AR) and Marginal Revenue (MR)

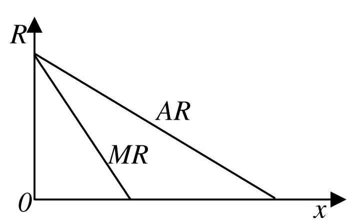
When average revenue curve slopes downwards, i.e., when AR is declining, its slope will be negative. Thus when AR curves slopes downwards MR curve will lay below AR curve. This is common in imperfect competitive market.

When AR curves slopes downwards MR curve will lay below AR curve.

When AR is always same then there is no difference between AR and MR. This relationship will be applicable for pure competitive market.



Pure competitive market



Imperfect competitive market

Illustrative Examples:

Example-1:

Let the unit demand function be $x = ap + b$ and the cost function be $c = ex + f$, where x = sales (in units), p = price (in Tk.), f = fixed cost (in Tk.), *e* = variable cost, *b* = demand, and when *p = 0, a* = slope of unit demand function.

Required: (i) find the cost *C* as a function of *p*.

(ii) find the revenue function *R (x)*.

(iii) find the profit function *P (x).*

Solution:

(i) Cost, $c = ex + f$

$$= e(ap + b) + f \ .$$

(ii) Revenue = price × quantity

$$= p \times x$$

$$= (\frac{x}{a} - \frac{b}{a}).x$$

$$= \frac{x^2}{a} - \frac{bx}{a}$$

(iii) Profit = Revenue – Cost

$$\text{P (x)} = (\frac{x^2}{a} - \frac{bx}{a}) - (ex + f)$$

$$= \frac{x^2}{a} - \frac{bx}{a} - ex - f$$

$$= \frac{x^2}{a} - (\frac{b}{a} + e)x - f \ .$$

Example-2:

Find the average cost and marginal cost if total cost

$$\text{TC} = 1000 + 100\text{q} - 10\text{q}^2 + \text{q}^3$$

Solution:

$$\text{Average cost (AC)} = \frac{\text{TC}}{\text{q}} = \frac{1000 + 100\text{q} - 10\text{q}^2 + \text{q}^3}{\text{q}}$$

$$= \frac{1000}{\text{q}} + 100 - 10\text{q} + \text{q}^2$$

$$\text{Marginal cost (MC)} = \frac{\text{d}}{\text{dq}}(\text{TC})$$

$$= \frac{\text{d}}{\text{dq}}(1000 + 100\text{q} - 10\text{q}^2 + \text{q}^3)$$

$$= 100 - 20\text{q} + 3\text{q}^2$$

Example-3:

If the demand function of the monopolist is $3q = 98 - 4p$ and average cost is $3q + 2$ where *q* is output and *p* is the price, find maximum profit of the monopolist.

Solution:

Given average cost, AC = $3q+2$

Total Cost, TC = AC × q = $(3q+2).\,q = 3q^2+2q$

Marginal cost (MC) = $\frac{d}{dq}(TC)$

$$MC = \frac{d}{dq}(3q^2+2q) = 6q+2$$

Again given that, $3q = 98-4p$

$$p = \frac{98-3q}{4}$$

Total Revenue (TR) = price × quantity

$$TR = p \times q$$

$$TR = \frac{98-3q}{4}.q = \frac{98q-3q^2}{4}$$

Marginal Revenue (MR) = $\frac{d}{dq}(\frac{98q-3q^2}{4})$

$$= \frac{98}{4}-\frac{6q}{4}$$

We know that under monopoly market, profit will be maximum at MC = MR.

$$6q+2 = \frac{98}{4}-\frac{6q}{4}$$

$$24q+8 = 98-6q$$

$$30q = 90$$

$$q = 3.$$

So the maximum profit of the monopolist will be obtained at q = 3.

Again Total Profit (TP) = TR – TC

$$TP = (\frac{98q-3q^2}{4}) - (3q^2+2q)$$

When q = 3, then the profit will be maximum

i.e., maximum profit = $\frac{(98)(3)-3(3)^2}{4} - 3(3)^2 - 2(3) = 33.75$

Example-4:

Show that marginal cost (MC) must equal marginal revenue (MR) at the profit maximizing level of output.

Solution:

We know that, Total profit = Total revenue – Total cost

i.e., TP = TR – TC

To maximize TP, $\frac{d(TP)}{dQ}$ must equal zero.

$$\frac{d(TP)}{dQ} = \frac{d(TR)}{dQ} - \frac{d(TC)}{dQ} = 0$$

$$\frac{d(TR)}{dQ} = \frac{d(TC)}{dQ}.$$

MR = MC (showed)

Example-5:

If the cost function is $C(x) = 4x + 9$ and the revenue function is $R(x) = 9x - x^2$, where x is the number of units produced (in thousands) and R and C are measured in million of Tk., find the following:

(i) Marginal revenue.
(ii) Marginal revenue at $x = 5$.
(iii) Marginal cost.
(iv) The fixed cost.
(v) The variable cost at $x = 5$.
(vi) The break-even point, that is $R(x) = C(x)$.
(vii) The profit function.
(viii) The most profitable output.
(ix) The maximum profit.
(x) The marginal revenue at the most profitable output.
(xi) The revenue at the most profitable output.
(xii) The variable cost at the most profitable output.

Solution:

Given that, $R(x) = 9x - x^2$

$C(x) = 4x + 9$

(i) $MR = \frac{d}{dx}(R) = \frac{d}{dx}(9x - x^2) = 9 - 2x$

(ii) When $x = 5$, then $MR = 9 - 2\times 5 = -1$.

(iii) $MC = \frac{d}{dx}(C) = \frac{d}{dx}(4x + 9) = 4$

(iv) The fixed cost, FC = 9.

(v) When $x = 5$, the variable cost (VC) is (4×5) = 20.

(vi) For break-even point, $R(x) = C(x)$.

$$9x - x^2 = 4x + 9$$

$$x^2 - 5x + 9 = 0$$

$$x = \frac{5 \pm \sqrt{-11}}{2}.$$

(vii) Profit, P = R – C = $\{ (9x - x^2) - (4x + 9) \} = 5x - x^2 - 9$.

(viii) Here profit, $P = 5x - x^2 - 9$

$$\frac{dP}{dx} = 5 - 2x$$

For maximum or minimum, $\frac{dP}{dx} = 0$

$$5 - 2x = 0$$

$$x = \frac{5}{2}.$$

Again $\frac{d^2P}{dx^2} = -2$; which is negative.

So the profit function is maximum at $x = \frac{5}{2}$.

Thus, the required most profitable output is $x = \frac{5}{2}$.

(ix) The maximum profit, $P = 5x - x^2 - 9$.

$$= 5 \times \frac{5}{2} - (\frac{5}{2})^2 - 9 \text{ ;}$$

$$= -\frac{11}{4}, \text{ which shows a loss.}$$

(x) When $x = \frac{5}{2}$, then $MR = 9 - 2x = 9 - 2 \times \frac{5}{2} = 4$.

(xi) When $x = \frac{5}{2}$, then $R = 9x - x^2 = 9 \times \frac{5}{2} - (\frac{5}{2})^2 = 16.25$.

(xii) When $x = \frac{5}{2}$, then variable cost $VC = 4x = 4 \times \frac{5}{2} = 10$.

Questions for Review:

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. The total cost *C* for output *x* is given by $C = \frac{2}{3}x + \frac{35}{2}$.

   Find (i) Total cost when output is 50 units.
   (ii) Average cost when output is 100 units.
   (iii) Marginal cost when output is 60 units.

2. The total cost of a firm is given by $C = 0.04q^3 - 0.9q^2 + 10q + 100$

   Find (i) Average cost
   (ii) Marginal cost
   (iii) Slope of average cost
   (iv) Slope of marginal cost
   (v) Value of *q* at which average variable cost is minimum.

3. The unit demand function is $x = \frac{25 - 2p}{3}$ where *x* is the number of units and p is the price. Assume that average cost per unit is Tk.50.

   Find (i) The revenue function R in terms of price *P*.
   (ii) The cost function *C*.
   (iii) The profit function *P*.
   (iv) The price per unit that maximizes the profit function
   (v) The maximum profit.

4. The marginal cost function (*y*) for production (*x*) is $y = 20 + 48x - 6x^2$ if the total cost of producing one unit is $50. Find the total cost function and average cost function.

5. The anticipated price of a product is Tk.15. The fixed cost of manufacturing the product is Tk.10,000 and the variable cost is Tk.10 per unit. Develop revenue, total cost function and calculate breakeven production. Calculate revenue, total cost, and profit at the breakeven product.

Lesson-2: Elasticity

After studying this lesson, you should be able to:

- Define the concept of elasticity, elasticity of demand, and elasticity of supply.
- Describe the techniques of measuring elasticity.

Elasticity

Elasticity is the ratio that measures the responsiveness or sensitiveness of a dependent variable to the changes in any of the independent variables. More clearly, the term elasticity refers to the percentage change in dependent variable divided by the percentage change in independent variable.

The term elasticity refers to the percentage change in dependent variable divided by the percentage change in independent variable.

Thus, elasticity = $\dfrac{\text{Percentage change in dependent variable}}{\text{Percentage change in independentvariable}}$

If $y = f(x)$, i.e., y depends on x, then the elasticity of y with respect to x is

Elasticity of y = $\dfrac{\text{Percentage change in } y}{\text{Percentage change in } x} = \dfrac{\%\ \Delta y}{\%\ \Delta x}$.

Elasticity of Demand

The elasticity of demand is the measure of responsiveness of demand for a commodity to the changes in any of its determinants. The determinants of demand are the commodity's own price, income, price of related goods (substitutes and complements) and consumers' expectations regarding future price.

The elasticity of demand is the measure of respon-siveness of demand for a commodity to the changes in any of its determinants.

$$Q_X^D = f(P_x, M, P_y, P_z ....)$$

where Q_X^D = quantity demanded of commodity x.

P_x = price of commodity x

M = money income of the consumer

P_y = price of the substitute, x and y are substitute to each other

P_z = price of complement, x and z are complement to each other.

Price Elasticity of Demand

The average price elasticity of demand is the proportionate response of quantity demanded to the change in price. Let δp be small change in price p and δx be a small change in the quantity demanded.

The average price elasticity of demand is the proportionate response of quantity demanded to the change in price.

The average price elasticity of demand = $\dfrac{\frac{\delta x}{x}}{\frac{\delta p}{p}} = \dfrac{p}{x}.\dfrac{\delta x}{\delta p}$

Since the point elasticity of demand is the limiting value of average price elasticity. So, the point elasticity of demand is $E_d = \frac{p}{x} \cdot \frac{dx}{dp}$.

Generally, slope of the demand curve is negative and thus E_d is negative, i.e.,

$$|E_d| = -\frac{p}{x} \cdot \frac{dx}{dp}.$$

Thus, when $|E_d| > 1$, the demand is elastic.

when $|E_d| < 1$, the demand is inelastic.

when $|E_d| = 1$, the demand is unitary elastic.

Price Elasticity of Supply

The price elasticity of supply measures the responsiveness of the quantity supplied of a commodity to a change in its price.

The price elasticity of supply measures the responsiveness of the quantity supplied of a commodity to a change in its price.

Price elasticity of supply = $\frac{\text{Percentage change in quantity supplied}}{\text{Percentage change in price}} = \frac{\% \, \Delta Q}{\% \, \Delta P}$

Thus, the price elasticity of supply is denoted by e_s and is given by $\frac{p}{x} \cdot \frac{dx}{dp}$; where x is quantity supplied and p is price.

Income Elasticity of Demand

The income elasticity of demand is a measure of the responsiveness of quantity demanded to a change in income.

The income elasticity of demand is a measure of the responsiveness of quantity demanded to a change in income, other things remaining the same. It is calculated by using the following formula:

Income elasticity of demand =

$$\frac{\text{Percentage change in quantity demanded}}{\text{Percentage change in income}}.$$

It is denoted by E_y and is given by $\frac{y}{x} \cdot \frac{dx}{dy}$ where x is the quantity demanded and y is the income per head in the relevant group of people.

The income elasticity of demand may be positive or negative. This motivates the definition of following types of goods.

Case 1: Goods are Luxury, if $E_y > 1$.

Case 2: Goods are Necessity of life, if $0 < E_y < 1$.

Case 3: Goods are Inferior, if $E_y < 0$.

Cross Elasticity of Demand

The cross elasticity of demand is a measure of the responsiveness of quantity demanded to the price of its substitutes or complements.

The cross elasticity of demand is a measure of the responsiveness of quantity demanded to the price of its substitutes or complements.

If x and y are related goods, the cross elasticity of demand,

$$E_{xy} = \frac{\frac{dQ_x}{Q_x}}{\frac{dP_y}{P_y}} = \frac{dQ_x}{dP_y} \cdot \frac{P_y}{Q_x}.$$

When x and y are complementary goods, the sign of the cross elasticity is negative and if x and y are substitute goods, then the sign is positive.

Illustrative Examples:

Example-1:

Find the elasticity of demand for the function $p = 100 - x - x^2$.

Solution:

$$\left|E_d\right| = -\frac{p}{x} \cdot \frac{dx}{dp}$$

$$= -\frac{p}{x} \cdot \frac{1}{\frac{dx}{dp}} \quad = -\frac{p}{x} \cdot \frac{1}{\text{-1-2}x}$$

$$= \frac{p}{x + 2x^2} \quad = \frac{100 - x - x^2}{x(1 + 2x)}$$

Example-2:

The demand function is $Q = 20 - 5P$. Find the inverse function and estimate the elasticity at $P = 2$.

Solution:

Given that $Q = 20 - 5P$.

The inverse function is $P = 4 - 0.2\,Q$

$$\frac{dQ}{dP} = -5$$

At P = 2, Q = 10.

$$\left|E_d\right| = -\frac{\text{dQ}}{\text{dP}} \cdot \frac{P}{Q} = -(-5) \cdot \frac{2}{10} = 1.$$

Example-3:

If the demand function is $p = 4 - 5x^2$; for what value of x, the elasticity of demand will be unity.

Solution:

The given demand function is $p = 4 - 5x^2$.

$$\frac{dp}{dx} = -10x.$$

The price elasticity of demand is $\left|E_d\right| = -\frac{p}{x}.\frac{dx}{dp}$

$$= -\frac{4-5x^2}{x}.\frac{-1}{10x} = \frac{4-5x^2}{10x^2}.$$

The elasticity of demand will be unity if $\left|E_d\right| = 1$.

$$\frac{4-5x^2}{10x^2} = 1.$$

$$10x^2 = 4-5x^2$$

$$15x^2 = 4$$

$$\therefore\ x = \frac{2}{\sqrt{15}}.$$

Example-4:

Find the elasticity of demand and supply at equilibrium price for demand function $p = \sqrt{100-x^2}$ and supply function $x = 2p - 10$, where p is price and x is quantity.

Solution:

Equilibrium conditions can be determined by equating demand and supply.

$$\sqrt{100-x^2} = \frac{x+10}{2}$$

or, $2\sqrt{100-x^2} = x+10$

or, $4\ (100 - x^2) = x^2 + 20x + 100$

or, $5x^2 + 20\text{x} - 300 = 0$

or, $x^2 + 4x - 60 = 0$

or, $(x + 10)\ (x - 6) = 0$

or, $x = 6$; (since negative quantity is not admissible).

$\therefore\ x = 8$.

Price elasticity of demand:

$$p = \sqrt{100-x^2}$$

$$\frac{dp}{dx} = \frac{1}{2}.(100-x)^{-1/2}.(-2x) = -\frac{x}{\sqrt{100-x^2}}$$

$$\left|E_d\right| = -\frac{p}{x}.\frac{dx}{dp} = -\frac{8}{6}.\frac{-x}{\sqrt{100-x^2}}$$

$$\therefore\ \left|E_d\right| = \frac{16}{9}$$

Price elasticity of supply:

Here $x = 2\text{p} - 10$

$$\frac{dx}{dp} = 2$$

Price elasticity of supply, $\text{E}_\text{S} = \frac{\text{p}}{\text{x}}.\frac{\text{dx}}{\text{dp}} = \frac{8}{6}.2 = \frac{8}{3}$

Questions for Review:

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. If the demand function is $Q = 1400 - P^2$. Find the price elasticity of demand at $P = 20$.
2. Find the elasticity of demand when demand function is $q = 7 - 2p$ at $p = 2$.
3. Find the elasticity of supply when supply function is $q = 2p^2 + 5$ at $p = 1$.

Lesson-3: Consumers' Surplus and Producers' Surplus

After completing this lesson, you will be able to:

- Explain consumers' surplus;
- Explain producers' surplus;
- Determine consumers' surplus;
- Determine producers' surplus.

Introduction

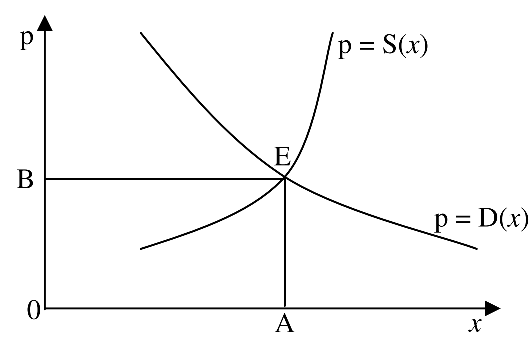
In the economic model of pure competition, it is assumed that all consumers of product pay the same price per unit of a product.

In the economic model of pure competition, it is assumed that all consumers of product pay the same price per unit of a product. This price comes about by the interplay of competitive market forces and is the price per unit at which the quantity of product consumers are willing and able to buy is matched by the quantity producers are willing and able to supply. The purpose of this lesson is to illustrate that this competitive situation benefits both consumer and supplier and to develop a measure of these benefits.

Market Equilibrium Position

Suppose the price p that a consumer is willing to pay for a quantity x of a particular commodity is governed by the demand curve $p = D(x)$. Further, suppose the price p that a producer is willing to charge for a commodity x of a particular commodity is governed by the supply curve $p = S(x)$. The point of intersection of the demand curve and the supply curve is called the equilibrium point E.

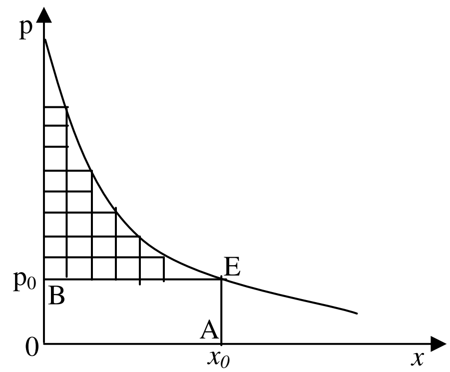
If the coordinates of E are (x_0, p_0), then p_0 is the market price a consumer is willing to pay for and a producer is willing to sell for a quantity x_0, the demand level of the commodity.



Consumer's Surplus (CS)

Difference between what consumers actually pay and the maximum amount that they would be willing to pay is called consumers' surplus.

A demand function represents the different prices consumers are willing to pay for different quantities of a good. In a free market economy, when some consumers' would be willing to pay more than the market equilibrium price for the commodity, this benefit to the consumers, i.e., difference between what consumers actually pay and the maximum amount that they would be willing to pay is called consumers' surplus.

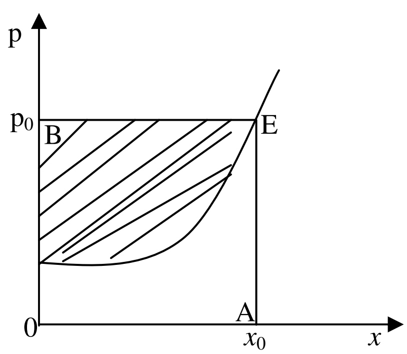


Thus, consumers' surplus = {total area under the demand curve *D(x)* from $x = 0$ to $x = x_0$} – {the area of the rectangle OAEB}.

Mathematically, Consumers' surplus = $\int_0^{x_0} D(x)dx - x_0 p_0$

Producer's Surplus (PS)

The relationship between the market price and the quantities producers are willing to supply is expressed as a supply function. The supply curve slopes upward to the right as because producers are willing to supply more at higher prices than at lower prices.



In a free market economy, when some producers would be willing to sell at a price below the market price p_0 that the consumers actually pays, the benefit of this to the producer, i.e., the difference between the revenue producers actually receive and what they have been willing to receive is known as producer's surplus (PS).

The difference between the revenue producers actually receive and what they have been willing to receive is known as producer's surplus (PS).

Thus, producers' surplus = {area of the rectangle OAEB} – {area below the supply curve from $x = 0$ to $x = x_0$}.

Mathematically, Producers' surplus = $x_0 p_0 - \int_0^{x_0} S(x)dx$

Illustrative Example:

Example-1:

The demand law for a commodity is $P = 1200 - q^2$. Find the consumer's surplus (CS) when the demand is15.

Solution:

Here given that $P = f(q) = 1200 - q^2$

If $q_0 = 15, P_0 = 1200 - 15^2 = 975$.

We know that consumer's surplus (CS) = $\int_0^{q_0} dq - p_0 q_0$

$$= \int_0^{15} (1200 - q^2)\text{dq} - 975 \times 15$$

$$= \left[1200q - \frac{q^3}{3}\right]_0^{15} - 14625.$$

$$= \left[1200 \times 15 - \frac{15^3}{3}\right] - 14625$$

= 2250.

Example-2:

The demand and supply functions under perfect competition for a product are $D(x) = 16 - x^2$ and $S(x) = 4 + x$ respectively. Find the market price, consumer's surplus (CS) and producer's surplus (PS).

Solution:

Here given that, demand function = $D(x) = 16 - x^2$ (i)

and supply function = $S(x) = 4 + x$ (ii)

Solving (i) and (ii) we get, $x = 3 = x_0$

When $x = 3$, $y = 7 = y_0$

We know that consumer's surplus (CS) = $\int_0^{x_0} D(x)dx - p_0 x_0$

$$= \int_0^{3} (16 - x^2)dx - 7 \times 3$$

= 18

Producer's surplus (PS) = $p_0 x_0 - \int_0^{x_0} S(x)dx$

$$= 7 \times 3 - \int_0^{3} (4 + x)dx$$

= 4.5

Example-3:

The demand and supply functions under perfect competition for a product are $D(x) = 16 - x^2$ and $S(x) = 2x^2 + 4$ respectively. Find the market price, consumer's surplus (CS) and producer's surplus (PS).

Solution:

Here given that, demand function $= D(x) = 16 - x^2$ (i)

and supply function $= S(x) = 2x^2 + 4$ (ii)

Solving (i) and (ii) we get, $x = 2 = x_0$

When $x = 2$, $y = 12 = y_0$

We know that consumer's surplus (CS) = $\int_0^{x_0} D(x)dx - p_0 x_0$

$$= \int_0^2 (16 - x^2)dx - 2X12$$

$= 5.33$

Producer's surplus (PS) = $p_0 x_0 - \int_0^{x_0} S(x)dx$

$$= 2\times 12 - \int_0^2 (2x^2 + 4)dx$$

$= 10.67.$

Example-4:

The demand and supply functions under pure competition for a product are $D(q) = 25 - q^2$ and $S(q) = 2q + 1$ respectively. Find the market price, consumer's surplus (CS) and producer's surplus (PS).

Solution:

For market equilibrium, Demand (D) = Supply (S).

$$25 - q^2 = 2q + 1$$

Here, q = 4, or –6

$\therefore$ $q_0 = 4$ (since q cannot be negative)

and $p_0 = 9$

We know that consumer's surplus (CS) = $\int_0^{q_0} D(q)dq - p_0 q_0$

$$= \int_0^4 (25 - q^2)dq - 4\times 9$$

$= 42.67$

Producer's surplus (PS) = $p_0 q_0 - \int_0^{q_0} S(q)dq$

$$= 4\times 9 - \int_0^4 (2q + 1)dq$$

$= 16$

Example-5:

The demand and supply functions under pure competition for a product are $D(x) = 113 - q^2$ and $S(x) = (q+1)^2$ respectively. Find the market price, consumer's surplus (CS) and producer's surplus (PS).

Solution:

For market equilibrium, Demand (D) = Supply (S).

$$113 - q^2 = (q+1)^2$$

Here q = 7, or –8

$q_0 = 7$ (since q cannot be negative)

$p_0 = 64$

We know that consumer's surplus (CS) = $\int_0^{q_0} D(q)\,dq - p_0 q_0$

$$= \int_0^{7} (113 - q^2)\,dq - 64 \times 7$$

= 228.67

Producer's surplus (PS) = $p_0 q_0 - \int_0^{q_0} S(q)\,dq$

$$= 64 \times 7 - \int_0^{7} (q+1)^2\,dq$$

= 277.67

Example-6:

Under a monopoly the quantity sold and market price are determined by the demand function. If the demand function for a profit-maximizing monopolist is $P(Q) = 274 - Q^2$ and $MC = 4 + 3Q$, find the consumer's surplus (CS).

Solution:

Given $P(Q) = 274 - Q^2$

TR = PQ = $(274 - Q^2) \times Q = 274Q - Q^3$

$MR = 274 - 3Q^2$

The monopolist maximize profit at MR = MC

$$274 - 3Q^2 = 4 + 3Q$$

$Q_0 = 9$ and $P_0 = 193$

We know that consumer's surplus (CS) = $\int_0^{Q_0} D(Q)\,dQ - P_0 Q_0$

$$= \int_0^{9} (274 - Q^2)\,dQ - 193 \times 9$$

= 486 units.

Example-7:

The demand and supply functions are $D(x) = (12 - 2x)^2$ and $S(x) = 56 + 4x$ respectively. Determine consumer's surplus (CS) under monopoly (so as to maximize the profit) and the supply function is identified with the marginal cost function.

Solution:

$$TR = x \times D(x) = x(12 - 2x)^2 = x(144 - 48x + 4x^2)$$
$$= 144x - 48x^2 + 4x^3$$

$$MR = 144 - 96x + 12x^2$$

Since supply price is identified with MC, we have

$$MC = 56 + 4x$$

In order to find consumer's surplus (CS) under monopoly, i.e., to maximize the profit,

we have $MR = MC$

$$144 - 96x + 12x^2 = 56 + 4x$$

or, $3x^2 - 25x + 22 = 0$

$\therefore$ $x = 1$ or $x = \frac{22}{3}$

When $x_0 = 1$, $D(x_0) = P_0 = (12 - 2)^2 = 100$

We know that consumer's surplus (CS) = $\int_0^{x_0} D(x)dx - p_0x_0$

$$= \int_0^1 (12 - 2x)^2 dx - 1 \times 100$$

$$= \frac{64}{3} \text{ units}$$

Again, when $x_0 = \frac{22}{3}, D(x_0) = P_0 = (12 - \frac{44}{3})^2 = \frac{64}{9}$

And consumer's surplus (CS) = $\int_0^{x_0} D(x)dx - p_0x_0$

$$= \int_0^{22/3} (12 - 2x)^2 dx - \frac{22}{3} \times \frac{64}{9}$$

$$= \frac{19360}{81} \text{ units.}$$

Questions for Review:

These questions are designed to help you assess how far you have understood and can apply the learning you have accomplished by answering (in written form) the following questions:

1. The demand law for a commodity is $P = 20 - D - D^2$. Find the consumer surplus when the demand is 3.

2. The supply and demand functions for a product are $S(x) = 3x + 9$ and $D(x) = 30 - 4x$ respectively, where x represents units of quantity. Compute consumer's surplus and producer's surplus.

3. Determine consumer's surplus and producer's surplus if $S(q) = 10 - \dfrac{5}{q+1}$ and $D(q) = 8 + \dfrac{15}{q+1}$.