



C7: Quantitative Techniques

Introduction and Module 1

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Contents

About this course manual	1
How this Course manual is structured	1
Course overview	3
Welcome to Quantitative Techniques	3
Course contents	4
C7: Quantitative Techniques — is this course for you?	4
Course outcomes	5
Timeframe	5
Study skills	6
Need help?	7
Activities	7
Assessments	7
Assessment schedule	8
Final examination	9
Getting around this Course manual	10
Margin icons	10
Module 1	11
Mathematical tools	11
Introduction	11
Unit 1	12
Basic operations with numbers	12
Introduction	12
Terminology	12
Number systems and operations	13
Activity 1.1	13
Activity 1.2	20
Activity 1.3	23
Sequences and series of numbers	24
Activity 1.4	25
Activity 1.5	28
Activity 1.6	29
Applications	29

Activity 1.7	31
Activity 1.8	32
Activity 1.9	35
Activity 1.10	36
Activity 1.11	37
Activity 1.12	39
Activity 1.13	40
Remember these key points	41
Unit summary	42
Unit 2	43
Linear equations and applications.....	43
Introduction	43
Linear equations and straight-line graphs	43
Activity 1.14	52
Activity 1.15	55
Activity 1.16	57
Applications.....	58
Activity 1.17	60
Activity 1.18	61
Activity 1.19	62
Activity 1.20	63
Remember these key points	64
Unit summary	65
Unit 3	67
Non-linear functions and equations	67
Introduction	67
Quadratics and their graphs.....	67
Activity 1.21	70
Activity 1.22	73
Exponentials, logarithms and their graphs	73
Activity 1.23	76
Apply non-linear functions.....	76
Activity 1.24	78
Remember these key points	79
Unit summary	80
Unit 4	81
Differentiation and integration.....	81
Introduction	81
Differentiation	81
Activity 1.25	84
Activity 1.26	85
Optimisation: Maxima and minima.....	86

Activity 1.27	89
Activity 1.28	90
Integration	91
Activity 1.29	92
Activity 1.30	93
Activity 1.31	95
Applications of differentiation and integration	95
Activity 1.32	98
Remember these key points	99
Unit summary	100
Assessment	101
<hr/>	
References	103
<hr/>	
Further reading	104
<hr/>	

About this course manual

How this Course manual is structured

The course overview

The course overview gives you a general introduction to the course. Information contained in the course overview will help you determine:

- If the course is suitable for you
- What you will already need to know
- What you can expect from the course
- How much time you will need to invest to complete the course.

The overview also provides guidance on:

- Study skills
- Where to get help
- Course assignments and assessments
- Activity icons
- Units.

We strongly recommend that you read the overview **carefully** before starting your study.

The course content

The course consists of five modules. Each module has four or five units and comprises:

- An introduction to the module content
- Module outcomes
- A list of the units in the module
- Unit outcomes
- New terminology, as applicable
- Core content of the unit with a variety of learning activities and case studies
- A unit summary
- Assignments and/or assessments, as applicable.

Resources

If you are interested in learning more about this subject, a list of additional resources (books, articles and/or websites) is provided at the end of each module.



Your comments

After completing Quantitative Techniques, please take a few moments to give us your feedback on any aspect of this course. Your feedback might include comments about:

- Course content and structure
- Course reading materials and resources
- Course assignments
- Course assessments
- Course duration
- Course support (assigned tutors, technical help, and so on)

Your constructive feedback will help us to improve and enhance this course.

Course overview

Welcome to Quantitative Techniques

The aim of this course is to teach you the skills and language of decision science, so that you can understand and interpret business situations in a quantitative framework.

“Quantitative” means using numbers to measure and describe situations and objects. The trend nowadays is to measure and quantify business and management as well, and to collect as much information as you can.

The challenge lies in interpreting the information collected and using it to make decisions and to build your company. After completing this course, you should be able to transform problems (numbers on a spreadsheet as well as problems involving decision-making) into strategies and opportunities.

Quantitative techniques provide the mathematical, statistical and operations-research foundations that enable informed decision making and management. This course will teach you to apply and interpret quantitative results and should equip you with the analytical skills you need to understand, formulate and solve business problems. You will develop a combination of theoretical and applied skills within a modelling approach, which is integrated with computer-based spreadsheet modelling.

The aim is not to make you an expert in mathematics or statistics, but to show you how to use information to make decisions. However, you will need to invest time and effort, as this course is challenging. In the process of acquiring skills, you should also take time to reflect on, interrogate and critique the subject matter.

This course fits into the broad aim of the MBA to provide students with:

- a dynamic framework of analytical skills and intellectual tools,
- breadth of knowledge across the disciplines of business and management, and
- understanding, confidence and the language to influence strategy at all levels of an organisation.

A course on using quantitative methods has an important place in business management. It integrates with the other courses within the MBA to provide a well-rounded multi-disciplinary experience.



Course contents

Quantitative Techniques comprises five modules:

Module 1: Mathematical tools

Module 2: Statistical tools and probability theory

Module 3: Spreadsheet modelling in management

Module 4: Modelling in management

Module 5: Decision making, risk and challenges

C7: Quantitative Techniques — is this course for you?

This course is intended for people who want to:

- **use** and **critique** the tools of mathematics, statistics and operations research in the business environment, because the worlds of business and management increasingly are being dealt with in quantitative terms;
- **master** basic mathematics and statistics to ensure they have the confidence to **use**, **understand** and **interpret** data analysis based on mathematics and probabilities;
- **develop** decision-making skills using techniques from operations research and decision science (such as linear programming, optimisation, forecasting and regression);
- **be introduced to** the electronic tools available in an age where challenges, changes and rapid responses are expected on a daily basis (focusing on spreadsheet modelling using either the licensed software Excel or free openware such as OpenOffice); and
- **develop an understanding** of the modelling framework and the modern decision-making approach to business management, in the context of highly sophisticated and challenging economic and financial markets.

Course outcomes



Outcomes

Upon completion of Quantitative Techniques you will be able to:

- **demonstrate** knowledge and understanding of the basic mathematical concepts applied in decision making and management;
- **analyse** data with the aid of statistical tools and basic probability theory;
- **use** electronic spreadsheets as a decision-making tool;
- **explain** the philosophy and methods of modelling;
- **apply** decision-making techniques such as linear programming, time series analysis, forecasting, regression and decision trees; and
- **reflect on** and **compare** risk management models and new theories in business and economics.

Timeframe



How long?

This course will take approximately 120 hours of formal study time and is expected to take 20 weeks to complete.

The course consists of 20 units, so you will need to master one unit each week. However, the units vary in length and degree of difficulty.

We recommend that you set aside extra time for self-study. Quantitative Techniques is a challenging course that requires hours of practice and reflection to master.



Study skills



Study Skills

As an adult learner, your approach to learning will differ from your school days – you will choose what you want to study, you will have professional and/or personal motivation for doing so, and you will most likely need to fit your study activities around other professional or domestic responsibilities.

Essentially, you will control your learning environment and, as a consequence, you will need to consider performance issues related to time management, goal setting, stress management, and so on. You may also need to reacquaint yourself with areas such as essay planning, coping with exams and using the Internet as a learning resource.

Your most significant considerations will be time and space – that is, the time you dedicate to your learning and the environment in which you study.

We recommend that you take time now, before starting your self-study, to familiarise yourself with the issues you may face. The Internet has a number of excellent resources, such as:

- <http://www.how-to-study.com/>

The “How to study” website is dedicated to study skills resources. You will find links to information about study preparation (with a list of the nine essentials for a study location) and strategies for taking notes, reading textbooks, using reference sources and managing test anxiety.

- <http://www.ucc.vt.edu/stdyhelp.html>

This is the website of Virginia Tech’s Division of Student Affairs, where you will find links to information about time scheduling (including a “Where does time go?” link), controlling your study environment, note taking, how to read essays for analysis, memory skills, a study skill checklist and basic concentration techniques.

- <http://www.howtostudy.org/resources.php>

Another “how to study” website with useful links to information about time management, efficient reading, questioning/listening/observing skills, “hands-on” learning, memory building, plus tips for staying motivated and developing a learning plan.

These links were active at the time of writing and are our suggestions to start you on your way. If you want to look for more, go to <http://www.google.com> and search “self-study basics”, “self-study tips”, “self-study skills” or similar phrases.

Need help?



Help

Is there a course web site address?

What is the course instructor's name? Where can s/he be located (office location and hours, telephone/fax number, e-mail address)?

Is there a teaching assistant for routine enquiries? Where can s/he be located (office location and hours, telephone/fax number, e-mail address)?

Is there a librarian/research assistant available? Where can s/he be located (office location and hours, telephone/fax number, e-mail address)?

Is there a learners' resource centre? Where is it located? What are the opening hours, telephone number, who is the resource centre manager, what is the manager's e-mail address)?

Who do learners contact for technical issues (computer problems, website access, etc.)

Activities



Activities

A number of activities are provided within the modules. Solve the problems yourself, then compare your effort with the answer provided. These activities are part of the formative assessment for this course.

There are also teacher-marked activities or assessments. Your lecturer will give you information about how they will deal with these.

Each module has one or more case studies. These can be challenging and take some time to complete.

Assessments



Assessments

Summative assessments are found at the end of each module. These assessments include questions that require you to discuss, justify, explain, compare, and so on. This means a numerical answer is not sufficient; explain what your answer means and provide the reasons for the statements you make.

Assessment schedule



Assessments

Date:	XX/XX/XXXX
Value:	10%
Format:	XXX
Units covered:	1 – 4 (Module 1)
Date:	XX/XX/XXXX
Value:	10%
Format:	XXX
Units covered:	5 – 8 (Module 2)
Date:	XX/XX/XXXX
Value:	10%
Format:	XXX
Units covered:	9 – 12 (Module 3)
Date:	XX/XX/XXXX
Value:	10%
Format:	XXX
Units covered:	13 – 16 (Module 4)
Date:	XX/XX/XXXX
Value:	10%
Format:	XXX
Units covered:	17 – 20 (Module 5)

Final examination



Assignments

There is one final exam that you must complete for this course. This is a written closed-book exam, with the use of a pocket calculator permitted. Access to Excel is not available during the exam.

Assignment 1

Due date: XX/XX/XXXX

Value: 50%

Format: XXX

Modules covered: Modules 1–5

Getting around this Course manual

Margin icons

Icons are used in the margins to help you find your way around the Course manual. The icons “signpost” a particular piece of text, a new task or a change in activity.

A complete set of the icons used in this course is shown below. You should familiarise yourself with them before starting your study.

			
Activity	Assessment	Assignment	Case study
			
Discussion	Group activity	Help	Note it!
			
Outcomes	Reading	Reflection	Study skills
			
Summary	Terminology	Time	Tip

Module 1

Mathematical tools

Introduction

In this module, you will review some important mathematical concepts to set the scene for the modules that follow. Some of the concepts are basic, while other material is more challenging. The idea is not to make you an expert in abstract mathematics, but to enable you to understand enough to make or evaluate decisions in a quantitative environment and to communicate those decisions with confidence.

Mathematics has its own language and symbols, and these are intended to clarify rather than to confuse.

Decision analysis is founded on a solid understanding of the mathematics and statistics covered in the first two modules of this course. The key to your success is practice and repetition, and you will benefit from investing time in this module.

If you are confident that you already have a good grounding in mathematics, skip the theory and go directly to the activities.

Upon completion of this module you will be able to:



Outcomes

- **use** and **apply** basic mathematical operations;
- **understand** sequences and series of numbers (with application to compound interest rates, present value, future value and internal rate of return);
- **solve** linear equations and simultaneous linear equations (with application to supply, demand, break-even and budget constraints);
- **understand** and **apply** non-linear functions; and
- **understand** the roles of differentiation and integration in modelling (with application to optimisation, elasticity, surplus, growth rate and marginal cost).



Unit 1

Basic operations with numbers

Introduction

The basic tools used in quantitative methods are the number systems and the operations that can be applied to them. You will revise the basic concepts and, upon completion of this unit, you will be able to:



Outcomes

- **explain** the different number systems;
- **apply** various arithmetic operations to numbers (such as percentages and ratios);
- **explain** the rounding-off procedure;
- **identify** arithmetic and geometric sequences; and
- **apply** sequences and series of numbers to business situations.

Terminology



Terminology

Natural or counting numbers	These are 1, 2, 3...
Integers	These include negative numbers and zero to allow for the representation of deficit, debit and loss: ...-3, -2, -1, 0, 1, 2, 3...
Inverse	The inverse of 2 is $\frac{1}{2}$ and the inverse of $\frac{3}{4}$ is $\frac{4}{3}$.
Product	Product is the quantity obtained by multiplication.
Numerator and denominator	In the expression $\frac{p}{q}$ the number p is the numerator and q is the denominator.
Exponents	Quantity representing the power to which a given number or expression is to be raised, usually expressed as a raised symbol beside the number or expression
Fractions or rational numbers	These allow for inverses, ratios, quotients and

percentages: any number in the form $\frac{p}{q}$ where p and q are integers (and q is not zero) is rational.

Number systems and operations

Numbers

Note: In mathematics, three dots ... are used to indicate that a sequence of numbers is infinite.

Fractions

Fraction examples include $\frac{1}{2}$, $\frac{2}{1}$, $\frac{3}{4}$, $\frac{-1001}{7}$, $\frac{245}{178}$ and so on.

Fractions should be simplified by dividing any common factor of the numerator and denominator into both:

$$\frac{10}{20} = \frac{5}{10} \text{ (because you divide by 2 on both the top and the bottom).}$$

$$\text{Then } \frac{5}{10} = \frac{1}{2} \text{ (because you divide by 5 on both top and bottom).}$$

Different fractions can be given a common denominator by first finding the product of the denominators and then expressing each fraction in terms of the new denominators.

Practise these skills with the examples provided in Activity 1.1.

Activity 1.1



Activity

Simplify rational numbers
and find common
denominators

What will you do?

Simplify the following rational numbers. Write your answers in the space provided in your workbook.

Make sure you complete the activity before you check the answers.

1. $\frac{16}{24}$
2. $\frac{35}{105}$
3. $\frac{-28}{200}$
4. $\frac{13}{28}$



Write the following numbers with a common denominator.

5. $\frac{3}{4}$ and $\frac{2}{3}$

6. $\frac{5}{12}$ and $\frac{1}{2}$

7. $\frac{2}{5}$, $\frac{5}{6}$ and $\frac{3}{2}$

Here's our feedback

1. $\frac{16}{24} = \frac{8}{12} = \frac{4}{6} = \frac{2}{3}$

2. $\frac{35}{105} = \frac{7}{21} = \frac{1}{3}$

3. $\frac{-28}{200} = \frac{-7}{50}$

4. $\frac{13}{28}$ (no common factors)

5. Use 12 (4×3) as the common denominator. Then:

$$\frac{3}{4} = \frac{9}{12} \text{ (because you multiply the top and bottom by 3)}$$

$$\text{and } \frac{2}{3} = \frac{8}{12} \text{ (multiply the top and bottom by 4).}$$

6. Use 24 (12×2) as the common denominator. Then:

$$\frac{5}{12} = \frac{10}{24} \text{ (multiply the top and bottom by 2)}$$

$$\text{and } \frac{1}{2} = \frac{12}{24} \text{ (multiply the top and bottom by 12).}$$

You can also use 12 as the common denominator.

$$\text{Then } \frac{5}{12} \text{ remains } \frac{5}{12} \text{ and } \frac{1}{2} = \frac{6}{12} \text{ (multiply top and bottom by 6).}$$

7. You can use 30 or 60 as a common denominator.

Decimal expansions

Fractions can be expressed as decimals and either as finite or infinite but recurring (with blocks of numbers that repeat indefinitely).

Examples of finite decimal expansions:

$$\frac{1}{2} = 0.5 \quad 2 = 2.0 \quad \frac{3}{4} = 0.75 \quad \frac{-1}{8} = -0.125$$

Examples of infinite decimal expansions:

- $\frac{-23}{3} = -7.666\dots$ (recurring digit 6)
- $\frac{3}{7} = 0.428571428571\dots$ (recurring block 428571)

Infinite recurring digits or blocks are indicated by a dot. In this way, the examples above are shown as $-7.\dot{6}$ and $0.\dot{4}2857\dot{1}$.

Similarly, $0.\dot{3}$ means $0.333333\dots$ and has the fractional form $\frac{1}{3}$.

In practice, recurring decimals and even finite but “long” decimals are always rounded off. This is done to a specified number of places and leads to an approximate number denoted by \approx . Your pocket calculator or computer software package will automatically do this by rounding off long or infinite expansions to a certain number of places.

Here are some examples, rounded off to four decimal places:

- $\frac{1}{3} \approx 0.3333$
- $-7.\dot{6} \approx -7.6667$
- $12.283357 \approx 12.2834$

These sets of numbers (natural, integer and rational) are all countable, infinite sets.

Rational numbers combine with the irrational numbers to form the real number system. Irrational numbers include π , e and $\sqrt{2}$. They form an uncountable infinite set and their decimal expansions are always infinite and non-recurring. There are no repeating blocks of digits in the decimal expansion of an irrational number. You calculate with them by using approximate rational numbers, which your pocket calculator or computer software package does automatically by rounding off to a certain number of places.

Here are some examples, rounded off as indicated:

- $\pi \approx 3.14159$ (to 5 decimals)
- $\sqrt{2} \approx 1.41421356$ (to 8 decimals)
- $e \approx 2.7183$ (to 4 decimals)

The real line

Real numbers can be represented as a continuum of points on an infinitely long line. To the right of zero lie all positive numbers, and to the left of zero lie all negative numbers, without limits. The positions of the whole numbers are usually indicated on the real line.

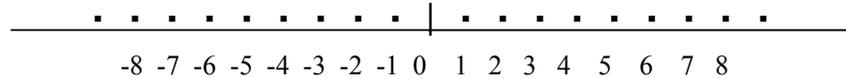


Figure 1 The real line

The numbers are ordered: $-7 < -6$ means -7 is less than and lies to the left of -6 ; and $8 > 7$ means 8 is greater than and lies to the right of 7 .

The real line can also be written as the interval $(-\infty, \infty)$ meaning the range or interval of all numbers. The symbol $-\infty$ is called minus (or negative) infinity and ∞ means plus (or positive) infinity.

Numbers can be restricted to domains or ranges. For example, $(-3, 56)$ means all real numbers between -3 and 56 and $(-\infty, 1)$ means all numbers smaller than 1 .

Note: In abstract (algebraic) language, numbers can be shown by symbols such as a , b , m and n .

Arithmetic operations

There are only two operations in arithmetic: addition and multiplication.

Subtraction means **adding** a negative number (debit or loss) and division means **multiplying** by the inverses.

In this way, $2 - 3$ means $2 + (-3)$. The value is -1 .

And $2 \div 3$ or $2/3$ means $2 \times \frac{1}{3}$ (the inverse of 3). The value (rounded off) is 0.667 .

The inverse of 12 is $\frac{1}{12}$ which can also be shown as $(12)^{-1}$

Note: Multiplication can be indicated in different ways:

$$2 \times 4 = 2(4) = 2*4$$

The following rules in the language of algebra use symbols to indicate numbers. You can substitute any number for symbol a below and the rule will hold.

Reminders about calculating with 0, 1 and -1:

- $a + 0 = a$ and $a - 0 = a$ but $0 - a = -a$ for any real number a
- $a \times 0 = 0 = 0 \times a$ for any real number a
- $0 \div a = 0$ **BUT** $a \div 0$ is not defined as a number
- $1 \times a = a \times 1 = a$ for any real number a
- $-1 \times a = a \times (-1) = -a$
- $-(-a) = +a = a$
- $(-a) \times (-a) = a \times a$
- $a + (-b) = a - b$ for any real numbers a and b
- $-a - (-b) = -a + b = b - a$

The distributive law says that: $a(b + c) = (a \times b) + (a \times c) = ab + ac$.

$$-a(b + c) = (-a \times b) + (-a \times c) = -ab - ac$$

$$-a(b - c) = (-a \times b) + (-a \times -c) = -ab + ac$$

Reminders about adding and multiplying fractions:

Fractions can only be added (or subtracted) if they have a common denominator:

$$\frac{2}{5} + \frac{5}{6} + \frac{3}{2} = \frac{12}{30} + \frac{25}{30} + \frac{45}{30} = \frac{82}{30}, \text{ which can be simplified to } \frac{41}{15}.$$

$$\text{Any fractions can be multiplied: } \frac{2}{5} \times \frac{5}{6} \times \frac{3}{2} = \frac{30}{60} = \frac{1}{2}$$

Exponentials

The notation for 2×2 is 2^2 or 2^2 (read as 2 to the power of 2 or 2 squared). The notation for $2 \times 2 \times 2 \times 2 \times 2$ is 2^5 or 2^5 (read as 2 to the power of 5).

In general, number a^n has base a (any real number) and exponent n (any integer).

Remember: Any number to the power of zero equals 1.
For example, $2^0 = 1$

Any negative number squared becomes a positive number:

$$(-1)^2 = (-1) \times (-1) = 1$$

$$\text{and } (-4.5)^2 = 20.25$$

Multiplication of exponentials with the same base is easy – you just add the exponents:

$$3^5 \times 3^2 = 3^{5+2} = 3^7$$

$$3^5 \div 3^2 = 3^5 \times 3^{-2} = 3^{5-2} = 3^3 = 27$$



In the decimal system, powers of 10 play an important role.

In this way, $1,000 = 10^3$, $100 = 10^2$, $10 = 10^1$, $1 = 10^0$, $0.1 = 10^{-1}$, $0.01 = 10^{-2}$ and so on.

This is used for what is known as “scientific notation”. Your pocket calculator may show numbers with only one digit before the decimal point – for example, 1.228 02 or 1.228 E 02 in place of 122.8. The meaning of 02 or E02 is to multiply by 10^2 . In this way, 1.228 02 means $1.228 \times 10^2 = 122.8$. Similarly 3.574 -03 means $3.574 \times 10^{-3} = 0.003574$.

Remember: For every positive power of 10, the decimal point moves one place to the right. For every negative power of 10, the decimal point moves one place to the left.

Remember: In mathematics, “per” means “divide” (form a ratio) and “of” indicates multiplication. Per cent means “per hundred”.

Here are some examples:

- Five per cent of 200 is calculated as:

$$5\% \text{ of } 200 = \frac{5}{100} \times 200 = 5 \times 2 = 10$$

- The ratio of three hundred to two thousand is:

$$\frac{300}{2,000} = \frac{3}{20} = 0.15 = 15\%$$

Ratios

These describe the value of one quantity relative to another quantity.

They are expressed as fractions $\frac{a}{b}$ or in the form $a:b$. Ratios play an

important role in economics and finance and are used to show:

- rate of return (yield)
- present value (PV)
- price-earnings ($P-E$)
- exchange rates
- debt-equity
- gross profit margin
- inventory turnover

Activity 1.2 has some examples and explanations.

Squares and square roots

You have seen that 4^2 means $4 \times 4 (= 16)$. This operation can be inverted by taking the square root or $\sqrt{\quad}$:

$$\sqrt{4^2} = \sqrt{16} = 4$$

In general, the square root of a number is the value that, multiplied by itself, gives back the original number. Most pocket calculators have a button for $\sqrt{\quad}$.

Here are some examples:

$$\sqrt{25} = 5 \text{ because } 5 \times 5 = 25$$

$$\sqrt{4} = 2 \text{ because } 2 \times 2 = 4$$

$$\sqrt{8} = 2.8284 \text{ (rounded off). Use your calculator to verify this.}$$

Note: The square root of a negative number does not exist in the real number system. This is because any real number answer, multiplied by itself, gives a positive number and therefore cannot give back the original negative number. However, there is an extension of the real numbers to complex numbers in which $\sqrt{-1}$ is shown by the symbol i .

Absolute values

The absolute value of a negative number is its positive counterpart. The notation is $|\quad|$. For example:

$$|-1| = 1 \qquad |-10.3| = 10.3 \qquad |43| = 43$$

Rounding off numbers

Calculators and computer software round off numbers to a certain number of decimal places. You should also round off answers where it is necessary or appropriate to the situation. For example, monetary amounts have two decimals and quantities such as numbers of items should be whole numbers.

Here are some examples:

1. USD 1,200.47231 should be rounded off to USD 1,200.47.
2. If the maximum number of items that can be fitted into a box is calculated by some formula to be 23.574, the answer should be 23.
3. If you need 55.47 litres of paint, you should buy 56 litres.

Factoring

Earlier you saw that $a(b + c) = (a \times b) + (a \times c) = ab + ac$. Long expressions can be written more compactly by applying the converse idea. If terms in an expression have common factors, these can be collected and “herded together”.

**Here are some examples:**

- $2a + 4b$ has the common factor 2 in each term.
So $2a + 4b = 2(a + 2b)$
- $(a + b) + r(a + b)$ has the common factor $(a + b)$ in each term.
So $(a + b) + r(a + b) = (a + b)(1 + r)$

Activity 1.2

**Activity****Do arithmetic operations****What will you do?**

Write your answers to these problems in the space provided in your workbook.

- a) $20\% = 20/100 = 0.? = ?/?$
 - b) 15% of \$200 = ? \times \$? = \$?
 - c) $\frac{3}{4}$ of $40\% = \frac{3}{4} \times 0.40 = 0.? = ?\%$
 - d) $\frac{3}{4} \div 0.2 = \frac{3}{4} \div \frac{2}{10} = \frac{3}{4} \times \frac{10}{2} = \frac{?}{?} = ?$
 - e) $1 = 100/100 = ?\%$
- If the amount \$1,200.00 includes sales tax of 10%, what was the original amount excluding tax?
- The ratio of defective items to quality items in a production line is 2:18. If 2,400 items were produced, how many defective items were there? What percentage of items was of good quality?
- The price to earnings ratio (P/E) of a company is 26. What does this mean?
- What is the rate of return per annum (p.a.) on an investment whose value increases from 2,000 to 2,750 over the year?
- The exchange rate between USD (United States dollars) and CAD (Canadian dollars) is USD 1 = CAD 1.204.
The exchange rate between USD and INR (Indian rupees) is USD 1 = INR 49.578.
 - a) Convert USD 10,000 to CAD.
 - b) Convert CAD 5,500 to USD.
 - c) Convert CAD 550 to INR.
- Factorise these expressions:
 - a) $P + 12Q + 5P$
 - b) $m(2 + k) - 6(2 + k) + (m - 6)$
- The discount rate is 12.5% p.a. The expected cash inflow from an initial investment of CAD 15.02 million at the start of the year is

CAD 19.34 million at the end of the year. What is the present value (*PV*) of the inflow? What is the net present value (*NPV*)? (These concepts are discussed in more detail in the “Applications” section of this unit.)

Here's our feedback

1. a) $20\% = 20/100 = 0.20 = 1/5$
 b) $15\% \text{ of } \$ 200 = 0.15 \times \$ 200 = \30
 c) $\frac{3}{4} \text{ of } 40\% = \frac{3}{4} \times 0.40 = 0.30 = 30\%$
 d) $\frac{3}{4} \div 0.2 = \frac{3}{4} \div \frac{2}{10} = \frac{3}{4} \times \frac{10}{2} = \frac{30}{8} = 3.75$
 e) $1 = 100/100 = 100\%$
2. The original plus tax gives the fraction $1 + 0.10 = 1.1$. Therefore the original amount is $\$1,200.00 \div 1.1 = \$1,090.91$.
3. The ratio means that for every 20 ($2 + 18$) items, 2 are defective. This means 2 in 20 or 1 in 10 items are defective. The number of defective items was $0.1 \times 2,400 = 240$, so the percentage of quality items is 90%.

$$4. P/E = \frac{\text{share value}}{\text{annual earnings per share}}$$

Intuitively, this means the investor is willing to pay \$26 for \$1 of share earnings. The annual earnings describe the net profit of the company per share.

5. $\frac{2,750 - 2,000}{2,000} = 0.0375 = 3.75\%$
6. a) $\text{USD } 10,000 = \text{CAD } (1.204 \times 10,000) = \text{CAD } 12,040.00$
 b) If $\text{USD } 1 = \text{CAD } 1.204$, then $\text{CAD } 1 = \text{USD } \frac{1}{1.204} = \text{USD } 0.8306$. This means that $\text{CAD } 5,500 = \text{USD } 0.8306 \times 5,500 = \text{USD } 4,568.30$.
 c) $\text{CAD } 1 = \text{USD } 0.8306 = \text{INR } 0.8306 \times 49.578 = \text{INR } 41.179$. This means that $\text{CAD } 550 = \text{INR } 41.179 \times 550 = 22,648.45$.
7. a) $P + 12Q + 5P = 6P + 12Q = 6(P + 2Q)$
 b) $m(2 + k) - 6(2 + k) + (m - 6) = (2 + k)(m - 6) + (m - 6)$
 $= (m - 6)(2 + k + 1)$
 $= (m - 6)(3 + k)$
8. $PV = \frac{19,340,000}{(1 + 0.125)} = \frac{19,340,000}{1.125} = 17,191,111$
 $= \text{CAD } 17.19 \text{ million (rounded off)}$



$$NPV = PV - \text{initial investment} = 17.19 - 15.02 = \text{CAD } 2.17 \text{ million}$$



Case study:
EduBooks

Consider the following financial statements for EduBooks, which employs 40 people. Figures are given in pounds (millions).
Then do Activity 1.3 at the end of the statements.

Balance Sheet for EduBooks			
December 2009			
Assets		Liabilities and Shareholders' Equity	
Current assets		Current liabilities	
Cash	10.355	Accounts payable	18.900
Inventories	22.561	Income taxes	1.250
Total current assets	32.916	Total current liabilities	20.150
Property and equipment	55.765	Long-term debt	15.200
Less depreciation	12.879		
Property and equipment, net	42.886	Total shareholders' equity	40.452
Total assets	75.802	Total liabilities and shareholders' equity	75.802

The balance sheet formula is: Assets = Liabilities + Shareholders' Equity

Note: If any of these terms are unfamiliar, use books or the Internet to research their meanings.

Income Statement for EduBooks		
For years ending 2005 and 2006		
	2005	2006
Net sales	*	*
Cost of sales	(2.900)	(3.485)
Gross income	15.255	18.990
Operating expenses	(2.800)	*
Income Statement for EduBooks		
For years ending 2005 and 2006		
Operating income	12.455	
Interest expense	(1.006)	(1.006)
Net profit before tax	*	*
Tax	*	*
Net income	*	*

Activity 1.3



Activity Calculate for Edubooks

What will you do?

Calculate the following for the company:

- Complete the income statement by filling in all cells marked *.
(Assume a constant tax rate over 2005–2006 of 22% on net profit.
For 2006, assume a 10% increase in operating expenses.)
- Investigate the company's liquidity by calculating the current ratio from the balance sheet.
- What percentage of total current assets is cash?
- What percentage of total liabilities does shareholder's equity form?
- What is the ratio of long-term debt to equity?
- Measure the personnel productivity of the company by calculating the revenue per employee.
- What is the increase or decrease in net income from 2005 to 2006?
- If the discount rate is 7.3% p.a. for 2009, what would the present value of the total assets have been on 1 January 2009?



Sequences and series of numbers

Sequences

A sequence of numbers is an infinitely long list of numbers. We are interested in sequences where the numbers follow a pattern or show a trend. Data may be presented as a sequence of observations, and in particular observations made at regular intervals (such as days). Later units discuss business forecasting as a management tool and look at time series analysis.

Here are some examples:

- a) 2, 4, 6, 8, 10...
- b) 2, 4, 8, 16, 32...

Each number in a sequence is called a **term** of the sequence and is indicated by the symbol T_n if it is in position number n (n takes on the values 1, 2, 3, 4 and so on). A sequence is therefore a list of numbers $T_1, T_2, T_3 \dots T_n \dots$. The list does not have an end. In practical situations you can work with finite or truncated sequences of numbers (data).

In example (a) above, $T_1 = 2$, $T_2 = 4$ and $T_5 = 10$. There is a clear pattern in the sequence: *Add 2 to get the next term*. So $T_{10} = 20$.

In example b), $T_1 = 2$, $T_2 = 4$ and $T_5 = 32$. There is a pattern in the sequence: *Multiply by 2 to get the next term*. So $T_{10} = 1,024$.

Types of sequence: Arithmetic or geometric

Two basic patterns are used:

1. Arithmetic sequence: Each term follows on from the previous one by adding a fixed number. This number is the constant difference d . So $T_n = T_{n-1} + d$. Example a) above is an arithmetic sequence. The formula for calculating terms in arithmetic sequences from the first term T_1 is: $T_n = T_1 + (n - 1)d$
2. Geometric sequence: Each term follows from the previous one by multiplying by a fixed number. This number is the constant ratio r . So $T_n = T_{n-1} r$. Example b) is a geometric sequence. The formula for calculating terms from the first term in this case is: $T_n = T_1 r^{(n-1)}$

Activity 1.4



Activity Identify sequences

What will you do?

- Identify these sequences as either arithmetic or geometric. Give the formula for the general term and calculate T_{100} in each case.
 - 1, 2, 3, 4, 5, 6...
 - 2, 4, 8, 16...
 - 4, 2, 1, $\frac{1}{2}$, $\frac{1}{4}$...
 - 3, 8, 13, 18...
- Write the first four terms of the sequences from the formulas for the general term. Identify the sequence.
 - $T_n = 2n - 1$, $n = 1, 2, 3, \dots$
 - $T_n = 2^{2n}$, $n = 1, 2, 3, \dots$

Here's our feedback

- This is an arithmetic sequence with $T_1 = 1$ and $d = 1$.

$$T_n = T_1 + (n - 1)d = 1 + (n - 1)1 = n$$

$$T_{100} = 100$$
 - This is a geometric sequence with $T_1 = 2$ and $r = 2$.

$$T_n = T_1 r^{(n-1)} = 2(2)^{(n-1)} = 2^{1+n-1} = 2^n$$

$$T_{100} = 2^{100}$$
 - This is a geometric sequence with $T_1 = 4$ and $r = \frac{1}{2}$.

$$T_n = T_1 r^{(n-1)} = 4(0.5)^{(n-1)}$$

$$T_{100} = 4(0.5)^{99} = 4(2)^{-99} = 2^2(2)^{-99} = (2)^{-97} \text{ (This is a very small number!)}$$
 - This is an arithmetic sequence with $T_1 = 3$ and $d = 5$.

$$T_n = T_1 + (n - 1)d = 3 + (n - 1)5 = 5n - 2$$

$$T_{100} = 5(100) - 2 = 498$$
- $T_1 = 1, T_2 = 3, T_3 = 5, T_4 = 7$
 This is an arithmetic sequence with $d = 2$.
 - $T_1 = 2^2 = 4, T_2 = 2^4 = 16, T_3 = 2^6 = 64, T_4 = 16$
 This is a geometric sequence with $r = 4$.



Other sequences

There are many sequences that display patterns other than arithmetic or geometric. In these sequences, it can be difficult to discover the formula for generating the terms and, in fact, there may not be a formula at all. A sequence may even be a random sequence. Sometimes you can identify a trend or cyclical component in your data.

Recursive sequences have terms that each depend on the value of the previous term(s). The first term must be given as a “seed” to enable all the other terms in the sequence to be generated.

Here are some examples:

1. The sequence 1, 4, 9, 16... is neither arithmetic nor geometric. If you assume there is a pattern, then you conclude that each term is a square and the formula is $T_n = n^2$, $n = 1, 2, 3, \dots$
2. The sequence $-1, 1, -1, 1, -1, 1, -1, 1, \dots$ is neither arithmetic nor geometric. If you assume there is a pattern, you conclude that $T_n = (-1)^n$, $n = 1, 2, 3, \dots$
3. The sequence with $T_n = \frac{1}{n}$, $n = 1, 2, 3, \dots$ is neither arithmetic nor geometric. The terms are $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots$
4. The sequence with terms T_n where $T_n = n$ -th digit in the decimal expansion of $\sqrt{2}$ has no pattern. The coin-tossing sequence with $T_n = 0$ if heads is flipped and $T_n = 1$ if tails is flipped also has no pattern and is, in fact, random.
5. The sequence with $T_n = 2T_{n-1} + 1$, $T_1 = 3$ is the recursive sequence 7, 15, 31, 63...
6. The most famous recursive sequence is the Fibonacci sequence. Leonardo of Pisa (Fibonacci), who lived in Italy in the 12th and 13th centuries, decided that the progression of rabbits being born, starting with one male and one female, would look like this: 1, 1, 2, 3, 5, 8, 13, 21, 34...

The formula here is $T_n = T_{n-1} + T_{n-2}$, $T_1 = 1$ and $T_2 = 1$.

This sequence has been applied in areas such as biology, art theory, computer design and even futures trading. Many of the applications are rather fanciful, and probably questionable.



Reflection

Looking for and identifying patterns in numbers (data) is an important part of analysis and decision making. However, be aware that the human mind is predisposed to find patterns even where there is little evidence for it.

Reflect on these quotes by David Brooks, published in *The New York Times* (28 October 2008):

The mind has...an overdeveloped sense of cause and effect which primes us to see purpose and design everywhere, even where there is none.

We may have a "...tendency to see data that confirm our prejudices more vividly than data that contradict them..."

Trends and limits for sequences

Trends and limits have a role in management and projective forecasting. The sequence in Example 1 of “Other sequences” has an increasing trend and we say the limit is $+\infty$.

The sequence in Example 2 above has a cyclical (oscillating) trend and no specific limit.

The sequence in Example 3 has a decreasing trend and the limit is 0 (even though no term of the sequence ever becomes zero).

Series

A series is the sum of terms of a sequence. We will consider finite or partial sums only. S_n denotes the sum of the first n terms of the sequence:
 $S_n = T_1 + T_2 + T_3 + \dots + T_n$

Here is an example:

The sequence 2, 4, 6, 8... has terms $T_n = 2n$ where $n = 1, 2, 3, \dots$

So $S_4 = 2 + 4 + 6 + 8 = 20$.

The Greek symbol \sum is used for summation. It is a useful shorthand notation for the sum of terms in mathematics.

$$\sum_{i=1}^n T_i = T_1 + T_2 + T_3 + \dots + T_n$$

The sum of the first n terms of an arithmetic sequence is:

$$\sum_{i=1}^n [T_1 + (i-1)d]$$

and the sum of the first n terms of a geometric sequence is: $\sum_{i=1}^n T_1 r^{i-1}$



Note it!

Formulas for sums of finite arithmetic and geometric series

Arithmetic sequence: $S_n = \frac{n}{2}[2T_1 + (n-1)d]$

Geometric series: $S_n = \frac{T_1(1-r^n)}{1-r} = \frac{T_1(r^n-1)}{r-1}$



Activity 1.5



Activity

Find T_n , T_{10} , and S_{10}

What will you do?

Find T_n , T_{10} and S_{10} for these sequences. Use the space provided in your workbook.

- 1, 3, 9, 27...
- 4, 8, 12, 16...

Note: You will first have to identify the type of sequence and decide which formula to use.

Here's our feedback

1. This is a geometric sequence with $T_1 = 1$ and $r = 3$.

$$T_n = T_1 r^{(n-1)} = 1(3)^{(n-1)}$$

$$T_{10} = (3)^9 = 19,683$$

The sum of the first 10 terms of the sequence is

$$S_{10} = \frac{3^{10} - 1}{3 - 1} = 29,524$$

2. This is an arithmetic sequence with $T_1 = 4$ and $d = 4$.

$$T_n = T_1 + (n - 1)d = 4 + (n - 1)4 = 4n$$

$$T_{10} = 4(10) = 40$$

$$S_{10} = \frac{10}{2}[2(4) + (9)4] = 220$$

Activity 1.6



Activity

Identify sequences and discuss trends

What will you do?

1. Write down the first six terms of each sequence from the formula for these general terms.

Discuss trends, if any, in the numbers.

a) $T_n = 2n - 5$

b) $T_n = (2n)^{-1}$

c) $T_n = 3T_{n-1} + 2$, with $T_1 = 1$

2. Identify the sequences, where possible, and give the expression for the general term in each of these cases.

Discuss any trends in the numbers.

a) 5, 8, 11, 14...

b) $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \dots$

c) 3, 9, 27, 81...

3. Sales data has been collected over 12 months and the pattern is forecast to continue into the future. (The numbers are in thousands.)

Sales data:

1.5, 3.5, 5.5, 7.5, 9.5, 11.5, 13.5, 15.5, 17.5, 19.5, 21.5, 23.5

- a) Determine a formula to describe the sales in a general month.

- b) Use it to predict the sales in month 48.

- c) Find the total of all sales after 60 months.

Applications

In this section, you will apply your knowledge of numbers, sequences and series to the evaluation of business projects and annuities. You will calculate future and present values of cash flows, determine profitability indices, compare projects, do amortising and make decisions.

Project analysis

An investment project is a proposed investment in a long-term asset – that is, an asset the organisation intends to use to yield a return over a period longer than a year. Examples of investment projects include purchasing buildings, building warehouses for storage and buying equipment.

The decision to be made is whether a project will be profitable or not, or which one to choose of competing, mutually exclusive projects.



The first step in analysing an investment project is to estimate its associated cash flows.

Assume that the initial outlay (initial investment) for a project occurs as an outflow at time 0, and that the subsequent cash flows are located at the last day of each period in the project's useful life. In this way, time 1 is the end of the first period, time 2 is the end of the second period, and so on. These times can be denoted by $t = 0$, $t = 1$, $t = 2$, and so on. Periods may be weeks, months or years.

Graphically, cash flows can be represented as follows:

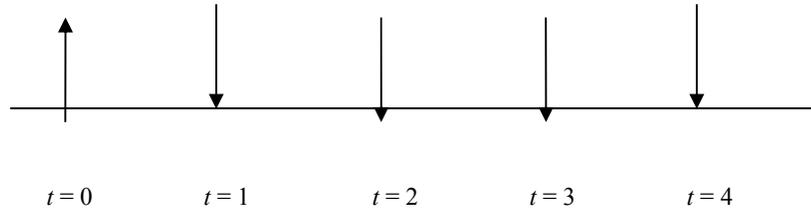


Figure 2 Graphical representation of cash flow

Cash outflows (↑) are negative and cash inflows (↓) are positive.

Money has a time value: to gain an objective view of the value of different cash flows for different projects, you need to compare values for a specific time. One way to do this is to evaluate all monetary values at time $t = 0$, where time 0 represents “now” or “the present”. This is called the **present value** of a cash flow and is obtained by discounting the cash amount by a discount rate. The discount rate is the required rate of return on the project and the maximum rate of return that can be earned on an investment of comparable risk.

At this stage, we will assume that there is no risk associated with the investment project and that all future cash flows are known with certainty. Then we may consider the interest rate of an investment in treasury bonds as the required rate of return, as this is considered an investment without risk.

Present value

Let C_t ($t = 1, 2, \dots$) be the cash flow at the end of period t and let r be the discount rate per period, expressed as a decimal. Then the present value (PV) or time 0-value of C_t is:

$$PV = \frac{C_t}{(1+r)^t}$$

Activity 1.7



Activity

Calculate present value

What will you do?

1. Assume the discount rate is 10% p.a. and that Project 1 promises you a cash inflow at the end of the year of 769,623 Ghanaian cedi (GHC 769,623). What is the present value?
2. If the discount rate is still 10% p.a. and the initial investment the same as for Project 1, but Project 2 offers you GHC 900,000 at the end of 2 years, what is its present value?
3. Which project looks more profitable?

Here's our feedback

1. $C_1 = 769,623$ and $r = 0.10$

$$PV = \frac{C_1}{(1+0.10)^1} = \frac{769,623}{1.1} = 699,657.27$$

2. $C_2 = 900,000$ and $r = 0.10$

$$PV = \frac{C_2}{(1+0.10)^2} = \frac{900,000}{(1.1)^2} = 743,801.65$$

3. Project 2 has a PV of GHC 743,801.65, which is larger than the PV from Project 1. However, you have to wait two years for your payout for Project 2. Your final decision will depend on your specific needs.



Reflection

Compound interest and discounting

Note that we work with compound interest rather than simple interest. Compound interest means that you receive interest on interest earned previously, whereas with simple interest you only earn interest on the initial investment.

When you invest amount A at compound interest rate r for two years, the value after one year is the initial amount plus the interest earned on it:

$$A + rA = A(1 + r)$$

The value at the end of year two is the new initial amount $A(1 + r)$ plus interest on it. So $A(1 + r) + rA(1 + r) = A(1 + r)(1 + r) = A(1 + r)^2$

In general, the final or future value (FV) after n years will be:

$$FV = A(1 + r)^n$$

Conversely, if the final value of an investment at the end of n years is given as amount C , we can find the initial or present value by solving for

$$A: A = PV = \frac{C}{(1 + r)^n}$$

This is what is meant by discounting.



Cash flow streams

If a project has an initial investment of value C_0 and yields cash inflow of value C_t at the end of each period t , then we can write the cash inflow stream as the sequence: $C_1, C_2, C_3, \dots, C_t, \dots$

Normally the project will have a finite life of length n , so that we will look at a finite or truncated sequence: $C_1, C_2, C_3, \dots, C_n$

The discounted cash flow stream is the sequence:

$$\frac{C_1}{(1+r)} \quad \frac{C_2}{(1+r)^2} \quad \frac{C_3}{(1+r)^3} \quad \dots \quad \frac{C_n}{(1+r)^n} \quad \dots$$

Net present value

You can evaluate a project over n years by calculating the net present value (NPV). It is the PV of all cash inflows, minus the initial investment.

$$NPV = \frac{C_1}{(1+r)} + \frac{C_2}{(1+r)^2} + \frac{C_3}{(1+r)^3} + \dots + \frac{C_n}{(1+r)^n} - C_0$$

If the NPV of a project is greater than zero, the present value of the project's total cash inflows is greater than the initial investment. Shareholder value should increase if the project is accepted. If $NPV = 0$, there is a break-even situation (you only get back what you put in). If you are looking at two or more mutually exclusive projects, your decision should be to accept the one with the largest NPV .

Activity 1.8



Activity

Analyse for investment

What will you do?

A company in Papua New Guinea has 30,000 PGK (kina) to invest in either Project A or Project B. Each project runs over three years. The expected cash inflows at the end of each year are displayed in the table.

Project A	Project B
Year 1: K 10,000	Year 1: K 14,000
Year 2: K 12,000	Year 2: K 12,000
Year 3: K 14,000	Year 3: K 10,000

Figure 3 Expected cash inflows

The required rate of return (discount rate) is 8% p.a.

1. Because both projects pay out K 36,000 over the three years, they appear to be equally profitable. Do an analysis to prove or disprove this.

- The *NPV* of Project A is:
 - The *NPV* of Project B is:
2. What conclusions can you draw? Are the projects equally profitable?

Here's our feedback

1. The *NPV* of Project A is:

$$\begin{aligned} NPV &= \frac{C_1}{(1+r)} + \frac{C_2}{(1+r)^2} + \frac{C_3}{(1+r)^3} - C_0 \\ &= \frac{10,000}{(1.08)} + \frac{12,000}{(1.08)^2} + \frac{14,000}{(1.08)^3} - 30,000 \\ &= 9,259.26 + 10,288.07 + 11,113.65 - 30,000 \\ &= 660.98 \end{aligned}$$

The *NPV* is greater than zero ($NPV > 0$) and investing in project A should lead to an increase in shareholder wealth.

The *NPV* of Project B is:

$$\begin{aligned} NPV &= \frac{C_1}{(1+r)} + \frac{C_2}{(1+r)^2} + \frac{C_3}{(1+r)^3} - C_0 \\ &= \frac{14,000}{(1.08)} + \frac{12,000}{(1.08)^2} + \frac{10,000}{(1.08)^3} - 30,000 \\ &= 12,962.96 + 10,288.07 + 7,938.32 - 30,000 \\ &= 1,189.35 \end{aligned}$$

The *NPV* is again greater than zero ($NPV > 0$) and investing in project B should also lead to an increase in shareholder wealth.

Project B has a greater *NPV* than Project A.

2. Project B is more profitable than Project A. This is because the cash inflows of B over the first two years are bigger than those of A, and re-investment of the cash will lead to better returns.

Profitability index

Another related measure of profitability is the profitability index (*PI*).

A project is accepted if the present value of net cash flows is greater than or equal to its initial investment.

$$PI = \frac{PV}{\text{Initial investment}}$$

Therefore the decision criterion for the profitability index is: If $PI \geq 1$, accept the project.

If mutually exclusive projects are being evaluated, the project with the largest profitability index is considered the best and is selected.



$$\begin{aligned}\text{For Project A: } PI &= \frac{30,660.98}{30,000} \\ &= 1.022\end{aligned}$$

Project A is profitable.

For Project B: $PI = 1.04$

Project B is more profitable than Project A.

At this stage, your decision should be to invest in Project B (bearing in mind that you have not looked at other factors such as risk). But what happens if the interest rate environment and economic situation change?



Reflection

Discount rate, risk and sensitivity analysis

Discount rate

How do you decide which discount rate to use in calculations? After the cash flows for an investment project have been estimated, the required rate of return on the project must be estimated. This is the discount rate. It may be considered the opportunity cost of capital and is the maximum rate of return that can be earned on an investment of comparable risk.

Risk

Under conditions of certainty, we assume there is no risk associated with the investment project and that all future cash flows are known with certainty. Then we may consider the interest rate of an investment in treasury bonds as the required rate of return, as this is considered an investment without risk.

However, remember that tax is payable on interest earned, which results in a reduction in the return yielded.

$$\text{Return rate after tax} = \text{Return rate before tax} \times (1 - \text{Tax rate})$$

When there is risk associated with an investment project, an investment with a comparable risk must be chosen and its expected rate of return is used as the required rate of return. Higher returns are usually required on more risky investments; therefore this approach involves adding a so-called “risk premium” to the rate of return on a risk-free investment.

Sensitivity analysis

A sensitivity analysis involves systematically asking “what if?” questions. Its objective is to determine how a decision will be affected by changes in inputs into the system. Doing sensitivity analysis is an important part of the decision-making process. When economic conditions change (inflation, increased interest rates and so on), the discount rate will change and a project may no longer be profitable.

Activity 1.9



Activity

Vary the discount rate

What will you do?

Using the information from Activity 1.8, do a sensitivity analysis by varying the discount rate to see whether the projects remain profitable under changed economic conditions.

Discuss the results.



Case study:
Decision on project investment

A company in Jamaica has 950,000 Jamaican dollars (JMD 950,000) to invest for four years. There are three mutually exclusive investment possibilities: A, B and C.

Investment A guarantees these cash inflows at the end of each of the four years:

Year	Investment A Cash inflows in JMD
1	300,000
2	400,000
3	500,000
4	600,000

Figure 4 Cash inflows for Investment A

Investment B is a four-year fixed investment which promises a simple interest rate of 12% p.a. The total value of the investment is paid out at the end of year four and there are no annual cash inflows at the end of years one to three.

Investment C is a treasury par bond with face value JMD 950,000, which gives the investor an annual cash inflow (coupon) of 6% p.a. The annual coupon is paid out at the end of each of the four years, and the initial investment or face value is paid back at the end of year four, together with the final coupon payment.



Activity 1.10



Activity

Decision on project investment

What will you do?

Analyse the possibilities from the case study and make a recommendation about which project the company should invest in.

Follow steps 1 to 5 to come to your decision.

1. Draw up a table for annual cash inflows for investments A, B and C.
2. Calculate the required rate of return (discount rate) for the three investments, assuming a tax rate of 30% p.a. on all investments and a return rate of 10% p.a. before tax.
3. Calculate the *NPV* and *PI* for the three investments.
4. Do a sensitivity analysis on the basis of possible future interest rate changes.
5. Write a report explaining your calculations and investment recommendation.

Annuities

The final application of operations on numbers, sequences and series that you will consider is annuities.

An annuity is a series of payments or receipts of equal amounts (R) of money each period for a specified number of periods. An ordinary annuity is one whose payments or receipts occur at the end of each period. The interest rate per period is denoted by i . The future value (FV) of the series of payments is:

$$FV = R(1+i)^{n-1} + R(1+i)^{n-2} + R(1+i)^{n-3} + \dots + R(1+i)^1 + R$$

$$= R + R(1+i)^1 + R(1+i)^2 + \dots + R(1+i)^{n-2} + R(1+i)^{n-1}$$

This is the sum of terms from a **geometric sequence** with $T_1 = R$ and ratio $r = (1+i)$.

Using the formula for the sum of a finite geometric sequence, the future value of an (ordinary) annuity is:

$$FV = R \frac{(1+i)^n - 1}{i}$$

The present value of an ordinary annuity can be expressed as:

$$PV = FV / (1+i)^n$$

$$\text{and as } PV = R \frac{(1+i)^n - 1}{i(1+i)^n}$$

These formulas can be put to good use. You can compute the fixed periodic payments R required on a loan to amortise or pay off the loan

over a given time period at a given interest rate. The amount of the loan is PV . For an ordinary annuity:

$$R = \frac{PV \times i \times (1+i)^n}{(1+i)^n - 1}$$

Or you may want to determine the annuity amount that must be deposited every year to accumulate to a certain lump sum FV . This type of arrangement is called a sinking fund. The formula for the future value of an annuity may be used to calculate this amount. For an ordinary annuity:

$$R = \frac{FV \times i}{(1+i)^n - 1}$$

We can also use the future value of an annuity formula to calculate the number of periods required for a series of equal payments to compound to a specific future value. In the same way, the formula for the present value of an annuity may be used to calculate the number of periods n required to amortise a loan with value PV when the payment R is given.

From the formula $PV = R \frac{(1+i)^n - 1}{i(1+i)^n}$ we see that $(1+i)^n = \frac{R}{R - iP}$

In Unit 3 of this module, you will see how to solve this equation for n but at this stage the result is stated as:

$$n = \frac{\ln R - \ln(R - iP)}{\ln(1+i)}, \text{ provided that } R > iP$$

The notation “ln” stands for the natural logarithm function. Any scientific or programmable pocket calculator will have an ln key that enables you to evaluate the formula above.

Activity 1.11



Activity

Calculate annuities

What will you do?

1. You pay a fixed amount of £100 per month at the end of each month for the next five years. The compound interest rate is 1% per month. How much money will you have saved at the end of the five years?
2. You want to accumulate USD 10,000 over three years at an interest rate of 4% p.a., by depositing a fixed amount at the end of every month. What should be the monthly amount deposited?
3. You plan to amortise a loan of INR 100,000 (rupees) by paying equal annual amounts of INR 20,000. The interest rate is 5.5% p.a. How long will it take you to pay off the loan?



Here's our feedback

1. This is an ordinary annuity with $R = 100$, $i = 0.01$ and $n = 5 \times 12 = 60$.

$$\text{The future value is } FV = 100 \frac{(1+0.01)^{60} - 1}{0.01} = 8,166.97 \text{ (pounds).}$$

2. $FV = 10,000$, $n = 3 \times 12 = 36$ and $i = \frac{0.04}{12} = 0.0033$ approximately.

Note that we are transforming the annual rate to an approximate monthly rate by dividing by 12.

$$\begin{aligned} R &= \frac{FV \times i}{(1+i)^n - 1} \\ &= \frac{10,000(0.0033)}{(1.0033)^{36} - 1} = \frac{33}{0.126} = 261.90 \end{aligned}$$

The monthly instalment should be USD 261.90.

$$\begin{aligned} 3. \quad n &= \frac{\ln R - \ln(R - iPV)}{\ln(1+i)} \\ &= \frac{\ln 20,000 - \ln(20,000 - 0.055(100,000))}{\ln(1.055)} \\ &= \frac{0.322}{0.0535} = 6.019 \end{aligned}$$

It will take about six years to pay off the loan.

Activity 1.12



Activity
Calculate annuities
yourself

What will you do?

1. You pay a fixed amount of \$50 per month at the end of each month for the next 10 years. The compound interest rate is 4% p.a. How much money will you have saved after 10 years?
2. What is the *PV* of annual payments of CAD 4,000 over five years at a rate of 3.4% p.a.?
3. A company in Pakistan wants to accumulate USD 10,000 over three years at an interest rate of 4% p.a. by depositing a fixed amount at the end of every month. Assume the exchange rate will stay fixed at USD 1 = PKR 80 (Pakistani rupees). What should the monthly amount be in PKR?
4. A loan of INR 50,000 must be amortised by paying equal monthly amounts of INR 2,000. The interest rate is 2.5% p.a. How long will it take to pay off the loan?



Activity 1.13



Activity
Understand the terminology

What will you do?

Use the following terminology table to record any words or terms you are uncertain about.

This activity is an opportunity to consolidate your understanding of any new terminology and concepts you encountered in Unit 1. Fill in the terms you have learned and write your own descriptions for them.



Terminology

Term	Description
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Remember these key points

- Natural or counting numbers are: 1, 2, 3...
- Integers are: ...-3, -2, -1, 0, 1, 2, 3...
- Fractions or rational numbers are any numbers in the form $\frac{p}{q}$ where p and q are integers (and q is not zero). They have finite or infinite but recurring decimal expansions. Numbers with infinite and non-recurring decimal expansions are called irrational. The union of rational and irrational numbers forms the real number system. The real line can be written as $(-\infty, \infty)$ meaning the range or interval of all numbers. The symbol $-\infty$ is called negative infinity and ∞ means positive infinity.
- Operations such as addition (including subtraction) and multiplication (including division) can be performed on numbers to solve problems. Numbers or quantities can be represented abstractly by symbols. This allows us to formulate problems in general terms, perform analyses and make decisions.
- A sequence of numbers is an infinitely long list of numbers, which are called **terms**. Sequences with a pattern or a trend are of particular interest.
- Arithmetic sequences have the general terms: $T_n = T_1 + (n - 1)d$
Geometric sequences have the general terms: $T_n = T_1 r^{(n-1)}$
- Sum of n terms of an arithmetic sequence are:

$$S_n = \frac{n}{2}[2T_1 + (n - 1)d]$$

- For a geometric series, sum of n terms are:

$$S_n = \frac{T_1(1 - r^n)}{1 - r} = \frac{T_1(r^n - 1)}{r - 1}$$



Unit summary

You have successfully completed this unit if you can:



Summary

- **understand** the different number systems: Natural numbers, integers, rational numbers and real numbers;
- **apply** various arithmetic operations to numbers;
- **identify** and **generate** arithmetic and geometric sequences;
- **understand** the rounding-off procedure;
- **find the sum** of finite series of numbers; and
- **understand** the application of operations on numbers, sequences and series of numbers to business situations: Cash flow, sensitivity analysis and annuities.

Unit 2

Linear equations and applications

Introduction

Algebra makes life easy. Its use of symbols allows you to cut through unnecessary information and display the essence of a situation. Symbols (such as x , y , t , z) take the place of quantities whose values are unknown or can change in different scenarios. These symbols are called **variables**. Variables can be related to each other via expressions such as equations. Equations can also contain given numbers or constants.

A relationship between variables x and y can generally be expressed by the equation $y = f(x)$, meaning that the value of y is a function of the value of x .

The function $f(x)$ can take on different forms, leading to different types of equations. Specific function values are denoted by substituting specific values for x . For example, $f(1)$ is the value of f when $x = 1$ and $f(0)$ is the value of f when $x = 0$, and so on.

Upon completion of this unit you will be able to:



Outcomes

- **identify** linear relations;
- **interpret** the slope (gradient) and intercepts;
- **draw** graphs of linear functions;
- **solve** linear equations and inequalities;
- **solve** simultaneous linear equations; and
- **apply** linear equations to management situations.

Linear equations and straight-line graphs

Linear relations $y = mx + c$

Linear equations are the simplest type of equation and describe or model situations where one quantity changes in proportion to another.

The relation $y = mx + c$ is the general form of a linear equation. Here, m and c are given constants, and y and x are the variables. In this form, we say that y is the dependent variable and x is the independent variable: as x is changed arbitrarily inside the right-hand expression “ $mx + c$ ”, the value of y is determined accordingly. We also write $y(x) = mx + c$ or sometimes $y = f(x) = mx + c$, meaning that y is a function of x , or dependent on x .

The relation of quantity y with quantity x in the expression $y = mx + c$ can be expressed graphically by a straight line. The graph is displayed on a set of labelled axes marked with the units of quantities. The independent variable is measured along the horizontal axis (x -axis) and the dependent



variable is displayed along the vertical axis (y -axis). The axes form a coordinate system that divides the plane into four quadrants. The graph itself consists of all points (x, y) that satisfy the linear relationship. The numbers x and y are also called **coordinates** of the particular point on the straight line. In Figure 5, the point $(2, 3)$ lies on the line $y = 0.5x + 2$ because the values $x = 2$ and $y = 3$ satisfy the equation.

The first quadrant consists of all points (x, y) such that $x > 0$ and $y > 0$.
 The second quadrant consists of all points (x, y) such that $x < 0$ and $y > 0$.
 The third quadrant consists of all points (x, y) such that $x < 0$ and $y < 0$.
 The fourth quadrant consists of all points (x, y) such that $x > 0$ and $y < 0$.

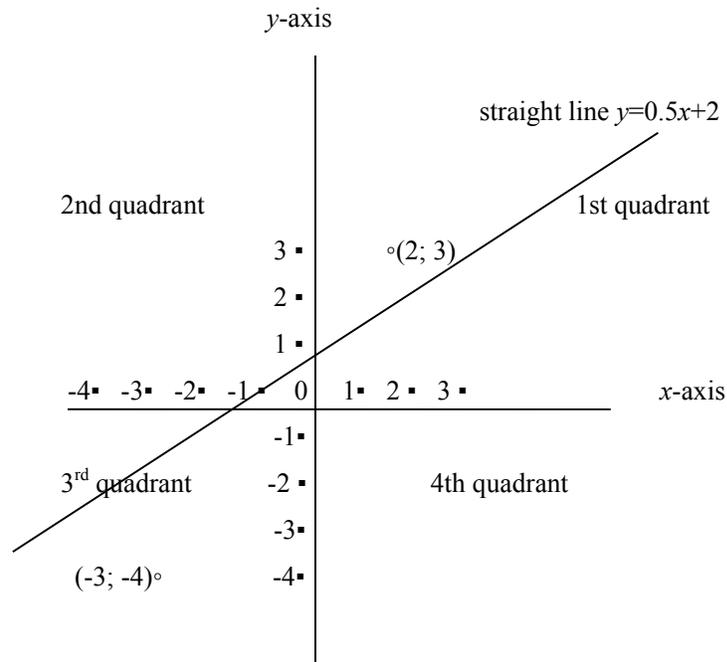


Figure 5 Graphical representation of linear equation.

Rate of change or slope

The rate of change of y with respect to x is determined by the number m . This is also the slope of the line. The slope can be determined if you have data points (x, y) . In fact, if you have two data points (x_1, y_1) and (x_2, y_2)

$$\text{then } m = \frac{y_2 - y_1}{x_2 - x_1}$$

In Figure 5, the slope is 0.5 or $\frac{1}{2}$. Start at the point $(0, 2)$ on the y -axis where the line intersects the axis. Move two units to the right and one unit up to reach $(2, 3)$ on the line.

This means $x_2 - x_1 = 2$ and $y_2 - y_1 = 1$, so that $m = \frac{1}{2} = 0.5$

The slopes of parallel lines are the same.

Intercept

The number c is the value of variable y when $x = 0$. It is called the y -intercept and indicates the point where the line cuts the vertical axis or y -axis. The x -intercept can also be determined. This is the value of x when $y = 0$ — that is, the point where the line cuts the horizontal axis or x -axis.

From the equation, we then have: $0 = mx + c$ or $mx = -c$. Solving for x gives intercept $x = \frac{-c}{m}$

Drawing straight-line graphs

1. If m and c are given, draw the line by marking off c on the vertical axis. Move one unit horizontally to the right and then m units vertically up (for positive m) or down (negative m). Mark this point and connect it to c with a straight line. Extend the line as needed.
2. Alternatively, choose two values for x (x_1 and x_2) and calculate the corresponding values (y_1 and y_2) from the given relation $y = mx + c$. Place the points (x_1, y_1) and (x_2, y_2) in the graph area of the coordinate system, connect them and extend the line.
3. If the relation between variables is linear and unknown, but a number (two or more) of data points (x, y) are given, plot them and connect with a straight line. To determine m , choose any two points (x_1, y_1) and (x_2, y_2) and calculate $m = \frac{y_2 - y_1}{x_2 - x_1}$
4. Computer software such as Excel or OpenOffice can be used to draw graphs. This is discussed in Module 3.

Solve linear equations

From $y = mx + c$ you can solve for various quantities.

1. If y , x and m are known, you can find c : $c = y - mx$
2. If y , x and c are known, you can find m : $m = \frac{y - c}{x}$ (provided $x \neq 0$)
3. If y , m and c are known, you can find x : $x = \frac{y - c}{m}$ (provided $m \neq 0$)
4. If two points (x_1, y_1) and (x_2, y_2) are given, you can find both m and c :

$$m = \frac{y_2 - y_1}{x_2 - x_1} \quad \text{and} \quad c = y_1 - \frac{y_2 - y_1}{x_2 - x_1} x_1 \quad \text{or} \quad c = y_2 - \frac{y_2 - y_1}{x_2 - x_1} x_2$$



Note it!

A quick primer on solving equations

1. An equation is a balance between the two sides of the = sign. What you do to one side, you must do to the other side.
2. To remove a number or variable from one side, you add (or multiply) its additive inverse (or multiplicative inverse) on *both* sides of the equation.
3. The aim is to repeat step 2 until you have the equation in the form where the unknown is on the left and the numerical value (answer) on the right.

Illustration: Solve for y in the equation $14 + 7x - 0.5y = 0$

Method: $14 + 7x - 0.5y = 0$

$$28 + 14x - y = 0 \quad (\text{Multiply both sides by } 2)$$

$$28 + 14x - y - 28 = 0 - 28 \quad (\text{Subtract } 28 \text{ from both sides})$$

$$14x - y = -28$$

$$14x - y - 14x = -28 - 14x \quad (\text{Subtract } 14x \text{ from both sides})$$

$$-y = -28 - 14x$$

$$y = 28 + 14x \quad (\text{Multiply both sides by } -1)$$

Draw graphs

Example 1: $y = 2x, x \geq 0$

This means y and x are in direct proportion ($m = 2$ and $c = 0$). As x increases (or decreases), y increases (or decreases) so that their ratio remains constant at 2. The slope is 2; the line goes through the origin of the axes so that both intercepts are zero.

The condition $x \geq 0$ indicates that x assumes only non-negative values. The graph is then restricted to the first quadrant of the coordinate system.

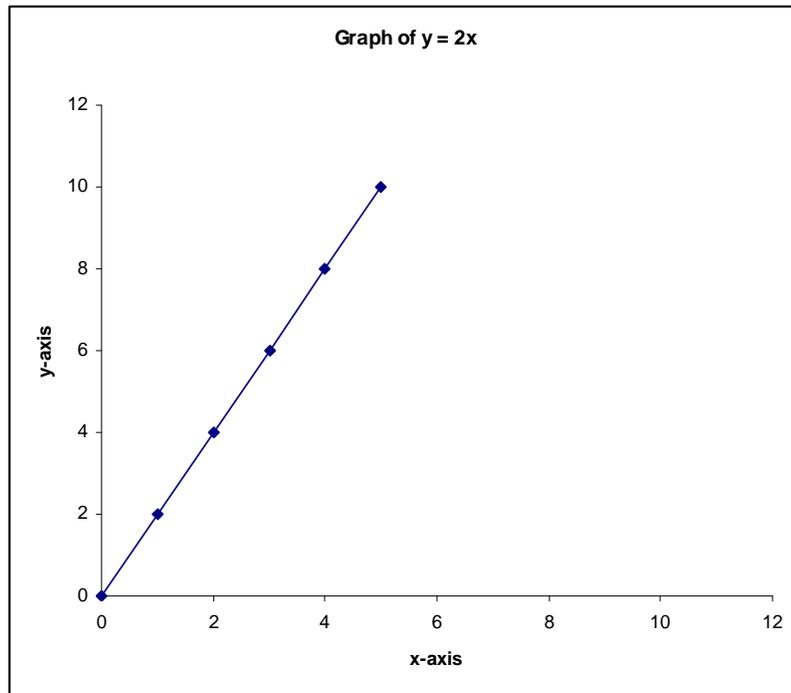


Figure 6 Graph of $y = 2x$

The line actually extends indefinitely upwards. Substitution of values for one variable gives values for the other variable: if $x = 3$ then $y = 6$. You can also write this as:

$$\text{If } y(x) = 2x, \text{ then } y(3) = 2(3) = 6$$

$$\text{If } y = 78 \text{ then } x = \frac{78}{2} = 39$$

**Example 2:** $y = 3, x \geq 0$

This means y remains constant at 3, no matter how x changes ($m = 0$ and $c = 3$). The rate of change of y with respect to x (slope of the line) is 0. There is no x -intercept.

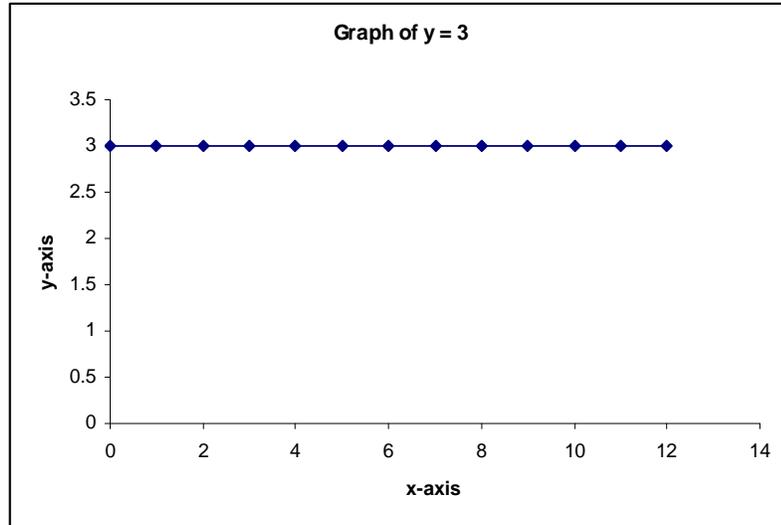


Figure 7 Graph of $y = 3$

The graph extends indefinitely to the right, parallel to the x -axis.

Example 3: $y(x) = 2x + 3, x \geq 0$ and $y \geq 0$

This means y increases or decreases as x increases or decreases respectively ($m = 2$), according to the formula. Relative change (slope) remains constant at 2. In fact, if x changes from x_1 to x_2 and y therefore changes from y_1 to y_2 , then the relative (rate of) change is:

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{2x_2 + 3 - 2x_1 - 3}{x_2 - x_1} = 2$$

The y -intercept is at $y = 3$ and the x -intercept is at $x = \frac{-3}{2}$

If $x = 3$, then $y = 2 \times 3 + 3 = 9$

If $y = 7$, then $2x + 3 = 7$

$$2x = 4$$

$$x = 2$$

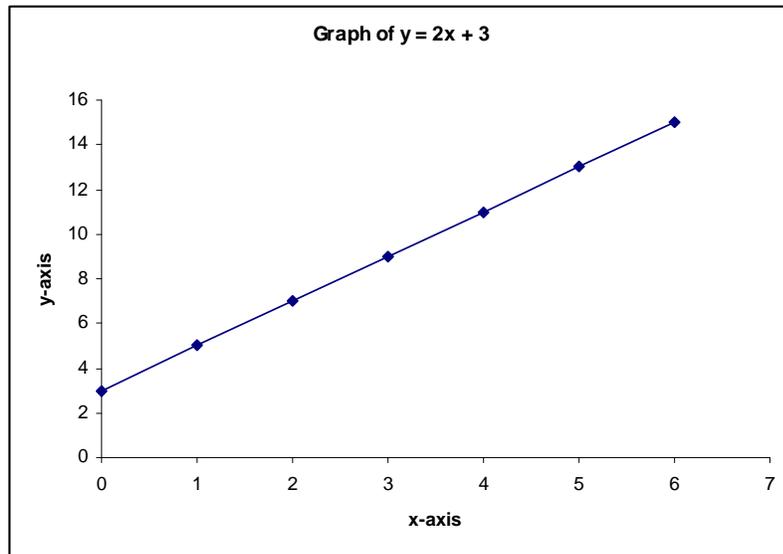


Figure 8 Graph of $y = 2x + 3$

The graph can be extended indefinitely to the right.



Note!
Warning

For convenience, only graphs for positive and finite values of the variables have been drawn. Some quantities can have negative values, in which case the axes and graphs are extended. Each situation will determine whether there are constraints on variables or not.

Specific data points are also indicated on the graphs in the preceding examples. These were used to draw the graphs in Excel. A similar situation will arise if you choose a number of points to help you plot the graph.

The examples at this stage are introductory and general. Specific applications to business, economics and management are discussed in the next sections.

Example 4: $2v + 4t - 26 = 0$ (t is the independent variable)

First, write this in standard form:

$$v + 2t - 13 = 0 \quad (\text{Divide by 2 on left and right})$$

$$v + 2t - 13 - 2t = 0 - 2t \quad (\text{Subtract } 2t \text{ on both sides})$$

$$v - 13 = -2t$$

$$v - 13 + 13 = -2t + 13 \quad (\text{Add 13 on both sides})$$

$$v = -2t + 13$$

Therefore $m = -2$ and $c = 13$

Now v decreases as t increases and vice versa. Relative change (rate of change or slope) remains constant at -2 .

The t -intercept is at $t = 7.5$.



As there is no restriction on t , we assume that t can take on negative values. In fact, the graph can be extended on the right and left, potentially infinitely.

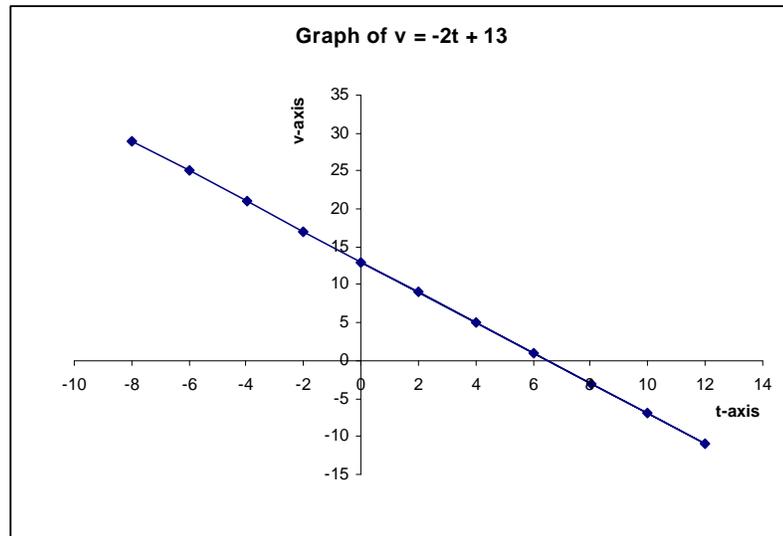


Figure 9 Graph of $v = -2t + 13$

What are linear inequalities?

Inequalities are expressions such as:

$$x > 3 \quad (x \text{ should be greater than } 3)$$

$$4y < 18 \quad (4y \text{ should be less than } 18)$$

$$y \geq x - 3 \quad (y \text{ should be greater than or equal to } x - 3)$$

Compare these expressions with the equalities:

$$x = 3 \quad (\text{a single point})$$

$$4y = 18 \quad (\text{a single point } y = 4.5)$$

$$y = x - 3 \quad (\text{straight line})$$

The solutions to the inequalities are not single points or lines, but sections of the real line or plane. Inequalities play an important part in linear programming (covered in Module 4).

Here are some examples:

- $x > 3$ is represented by the section of the real line to the right of point $x = 3$.

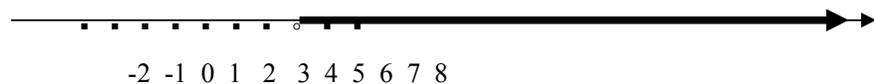


Figure 10 The real line for $x > 3$

2. $4y < 18$ is represented by the section of the line below point $y = 4.5$.

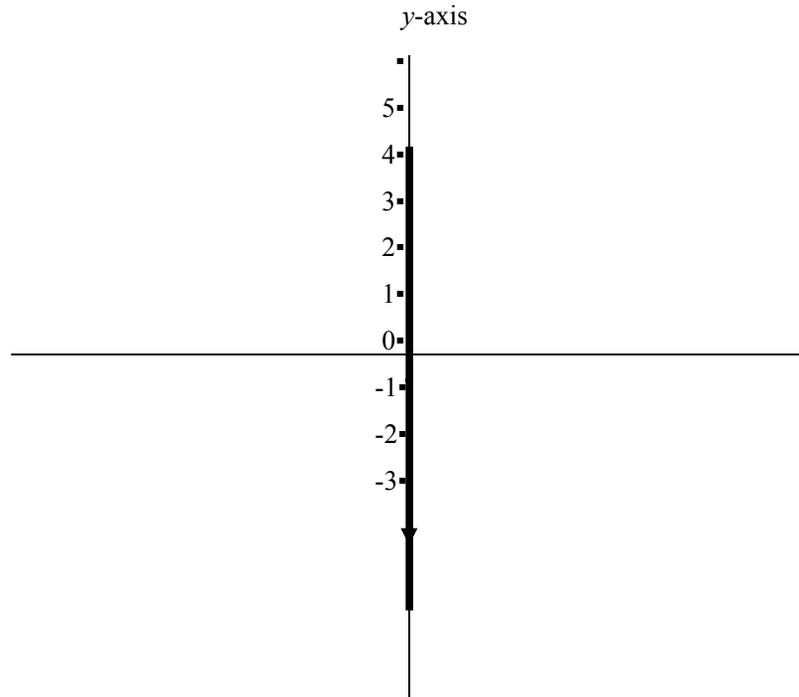


Figure 11 The real line for $4y < 18$

Inequality $-2 \leq x \leq 4$ is represented by a section or domain of the real line. It describes all numbers between and including -2 and 4 . We denote it by $[-2, 4]$.

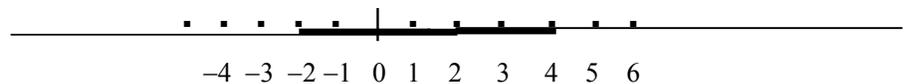
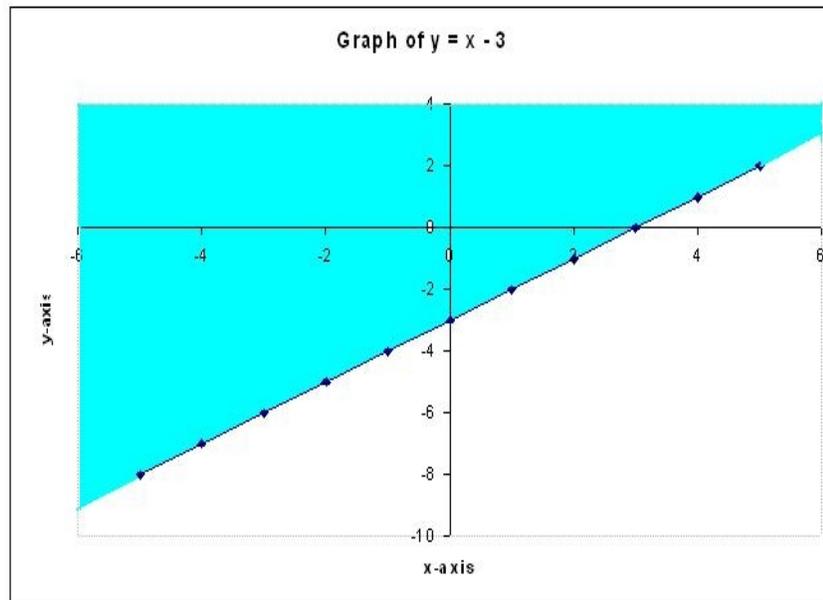


Figure 12 The real line for $-2 \leq x \leq 4$

3. Inequality $y \geq x - 3$ is represented by the blue region or area in the (x, y) - plane, from the line $y = x - 3$ upwards. The area below the line is described by $y < x - 3$.

The easiest way to decide whether the required area lies above or below the line graph is to substitute $(0, 0)$ into the inequality. If the inequality is true, then $(0, 0)$ must lie inside the required region. In this case, it is true that $0 \geq 0 - 3$, so the region $y \geq x - 3$ must include $(0, 0)$.

Figure 13 Graph of $y = x - 3$ 

Tip

Important! When you multiply by a negative number throughout an inequality, the inequality switches. In this way, $x > 8$ implies $-x < -8$. Substitute numbers for x to check this.

In a similar way, $-5y \leq 10x + 20$ implies $5y \geq -10x - 20$ or $y \geq -2x - 4$

Activity 1.14



Activity

Draw graphs

What will you do?

1. Draw the graph of the linear equation $P(Q) = 20 - 3Q$ with $P > 0$ and $Q > 0$.
2. Determine the slope and y -intercept of the line $y = 4x - 6$.
3. Draw the graph and give the equation for the linear relationship between quantities z and k if you are given data points $(2, 4)$ and $(4, 1)$. Take z as the dependent variable.
4. Represent the inequality $9x - 3y \geq -6$ graphically.

Here's our feedback

1.

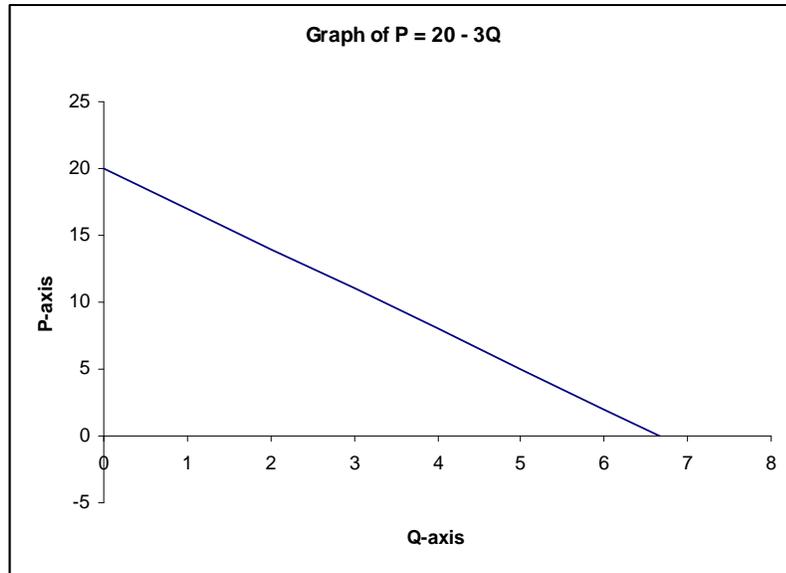


Figure 14 Graph of $P = 20 - 3Q$

2. Slope $m = 4$, y -intercept $= -6$

3. Calculate slope $m = \frac{1 - 4}{4 - 2} = \frac{-3}{2} = -1.5$

The equation is $z = -1.5k + c$

To find c , substitute one of the data pairs into the equation and solve for c . Using data point (2, 4) you get $4 = (-1.5 \times 2) + c$ and $c = 4 + 3 = 7$.

The equation is $z = -1.5k + 7$

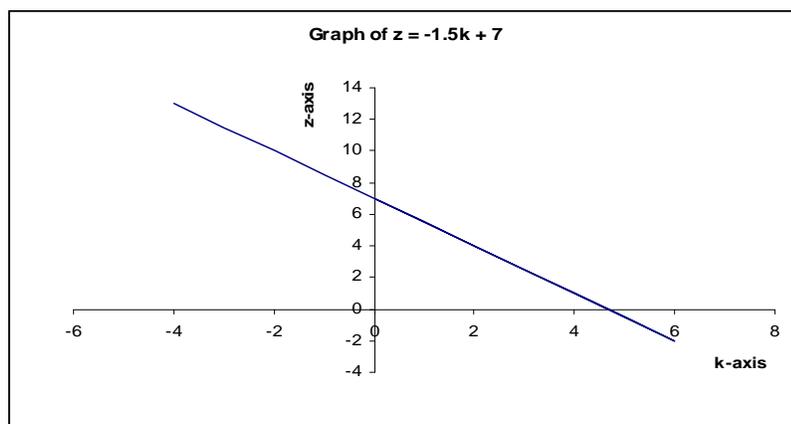


Figure 15 Graph of $z = -1.5k + 7$



4. First, write $3y \leq 6 + 9x$ in standard form — that is, $y \leq 2 + 3x$.
Next, draw the line $y = 3x + 2$.

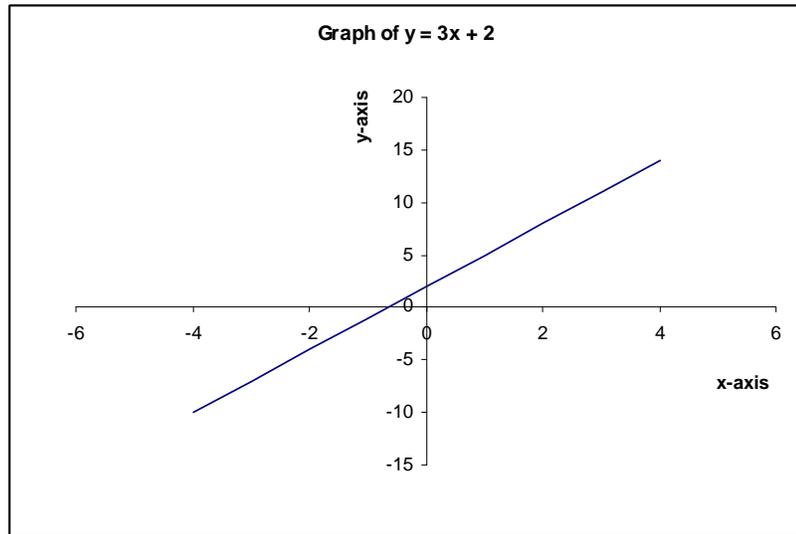


Figure 16 The line $y = 3x + 2$

Test whether $(0, 0)$ satisfies $y \leq 3x + 2$. You can see that $(0, 0)$ does lie in the required region, which is therefore below the line $y = 3x + 2$. You can shade in that area.



Tip

More variables?

There can also be linear equations with three or more variables. For example, in $z = 4x - y + 4$ the variable z is now dependent on independent variables x and y . You can also write this as $f(x, y) = 4x - y + 4$. For each pair of numbers (x, y) , apply function f and get the value for z or $f(x, y)$.

Graphically, this represents a flat plane in three-dimensional or (x, y, z) -space. A linear relation in four or more dimensions cannot be readily visualised.

Activity 1.15



Activity Draw graphs

What will you do?

1. Draw the graph of the equation $G = 5 - 6H$ under the constraint $H \geq 0$.
2. Determine the slope and all intercepts of the line $y = 4x - 6$.
3. Draw the graph of $y = x + 3$.
4. Draw the graph and give the equation for the linear relationship between quantities x and y if you are given data points $(-2, -1)$ and $(3, 5)$. Take y as the dependent variable.
5. You are given the data points $(-1, 3)$, $(3, 6)$ and $(5, 9)$. Without plotting points, determine whether there is a linear relation between the points. Now plot the points and substantiate your answer. Discuss.
6. Write the following equations in standard form ($y = mx + c$):
 - a) $4x + y - 9 = 0$
 - b) $3y + 12x = 8$
7. Solve each of these inequalities by representing it as a shaded region in the (x, y) -plane:
 - a) $x - 2y \leq 3$
 - b) $3y + x + 1 \leq y - 6$

Simultaneous equations

In this section, you will consider scenarios where two linear relations can apply between quantities. You will then find the specific data point that satisfies both relations.

Here is an example:

Consider the two quantities Labour force L and Wages W . A labour force for a company is balanced when the wages W_d demanded by workers equals the wages W_s supplied or afforded by the company. In turn, the two wages are linearly related to the labour force L_s supplied by workers and the labour force L_d demanded by the company, respectively. The relations are expressed by linear equations or functions f_d and f_s . So you have $W_d = f_d(L_d)$ and $W_s = f_s(L_s)$.

Each equation has its own parameters in terms of slope and intercept, and can be represented by a straight-line graph in the (L, W) coordinate system. The point of balance is where the two lines intersect and where labour demanded equals labour supplied: $L_d = L_s$. You will then also have agreement in wages: $W_d = W_s$. This is a unique point (unless the lines are parallel!) and represents equilibrium or balance for wages and size of workforce.



Algebraically, set $W_d = W_s = W$ and $L_d = L_s = L$. Then solve the equation $f_d(L) = f_s(L)$ for L . Substitute this value in either of the two equations and solve for W . You will have found the equilibrium wage and size of workforce. (See Model 3 in the next section on “Applications”.)

Solve simultaneous equations

Look now at the two general methods for solving simultaneous equations: the graphic method and the algebraic method.

Graphically, all you need to do is draw the two lines and find the point of intersection. However, the algebraic method may be faster and more accurate. (**Note:** At this stage, we will not consider the case of three or more simultaneous equations in three or more variables.)

Here are some examples:

Solve these equations simultaneously:

$$y = 2x - 6 \text{ and } y = 6x + 2$$

Algebraic method: We must have $2x - 6 = 6x + 2$

$$\text{So: } -6 - 2 = 6x - 2x$$

$$4x = -8$$

$$x = -2$$

The corresponding value for y is $y = 2(-2) - 6 = -10$.

The unique solution is the point $(-2, -10)$.

Graphic method: We draw both graphs in the (x, y) plane. The point of intersection is $(-2, -10)$.

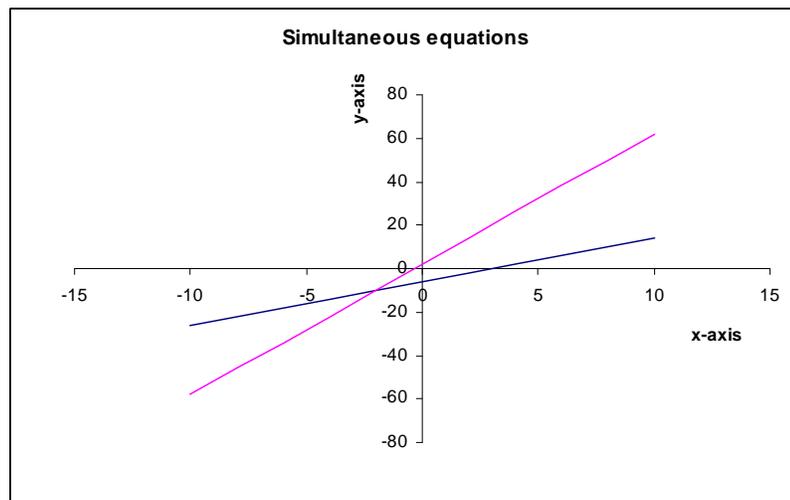


Figure 17 Graph of simultaneous equations

The algebraic method can take on different forms. You can write both linear equations in the form $y = mx + c$, equate the expressions and solve for x , then for y . Or you can solve for one variable in terms of the other, substitute into the remaining equation and solve.

Here is an example:

Solve simultaneously by algebraic methods

$$3x + 2y = 19 \text{ and } -12x + 6y = 3$$

Method 1: First, get the equations into the form $y = mx + c$.

$$y = \frac{-3}{2}x + 9.5 \text{ and } y = \frac{12}{6}x + \frac{3}{6} = 2x + 0.5$$

Equate the two expressions and solve:

$$\frac{-3}{2}x + 9.5 = 2x + 0.5$$

$$3.5x = 9$$

$$x = 2.57 \text{ (rounded off)}$$

The corresponding value for y is $y = 2(2.57) + 0.5 = 5.64$.**Method 2:** Solve for one variable in terms of the other from one equation; then substitute this into the second equation.From $-12x + 6y = 3$ you get $y = 2x + 0.5$, and substituting into the other equation you get: $3x + 2(2x + 0.5) = 19$

$$7x = 18$$

$$x = 2.57$$

You solve for y as before. Note that you can first solve for y and then for x .

Activity 1.16

**Activity****Solve simultaneous equations****What will you do?**

- Solve these simultaneous equations both graphically and algebraically:
 - $y = -x + 5, y = 5x - 7$
 - $3y - x + 5 = 0, 2y = 2x - 1$
- Discuss the possibility of solving these simultaneous equations:
 $y = -5x + 3.5, 2y + 10x + 6 = 0$
- Discuss the possibility of solving these simultaneous equations:
 $3y = x + 9, -12y + 4x = -36$
- Investigate whether it is possible to find a solution to these simultaneous equations:
 $y = -3x + 2, y = x - 4, 2y = -x + 5$
- Summarise the results of questions 1–4.



- Describe, in terms of graphs, the implications of solving equations in three variables. You do not have to draw the actual graphs, but try to imagine them in space.

Applications

As mentioned earlier, algebra is a tool for analysing and solving a variety of problems in areas such as physics, business, economics, finance, management, biology, medicine and so on.

In this section, we consider situations that can be modelled by linear equations. It is important to understand the role of mathematical models. These are approximate descriptions of real-life situations that help us to understand and analyse information. They are not exact reflections of reality but are set up under various simplified assumptions, some of which may be subjective. Nevertheless, if used correctly, models are a valuable and often indispensable tool in decision making.

Model 1: Supply and demand

Let Q denote the quantity of a certain product demanded by consumers and P the price of one item of the product. A simple form of the law of demand states that the higher the price, the less demand there will be for the product. Alternatively, this can be stated as: the more demand there is, the lower the price should become.

Assume that the relation between quantity Q of bicycles demanded by consumers from Company A and the price P charged is given by:

$$P(Q) = 2,000 - 0.2Q$$

Price P is a function of the quantity Q of items. The units will be in the currency of the country, such as pounds or dollars.

Various assumptions will have been made, for example that the income of consumers remains constant over the period under consideration, that competition from other companies remains the same and so on.

There are constraints in this case: P and Q should remain positive. In the form above, we consider P to be the dependent and Q to be the independent variable.

The intercept value of 2,000 indicates that as demand Q drops to zero, the price per bicycle increases to 2,000 pounds.

The other intercept is where $P = 0$, which gives $Q = 10,000$. The implication is that if the demand rises to close to 10,000 bicycles, the price should (theoretically) drop to zero.

The negative slope shows that as more bicycles are demanded, the company can drop the price. The value of 0.2 means that for every extra bicycle demanded, the price should drop by 0.2 (one-fifth) of the unit of currency.

The graph looks like this:

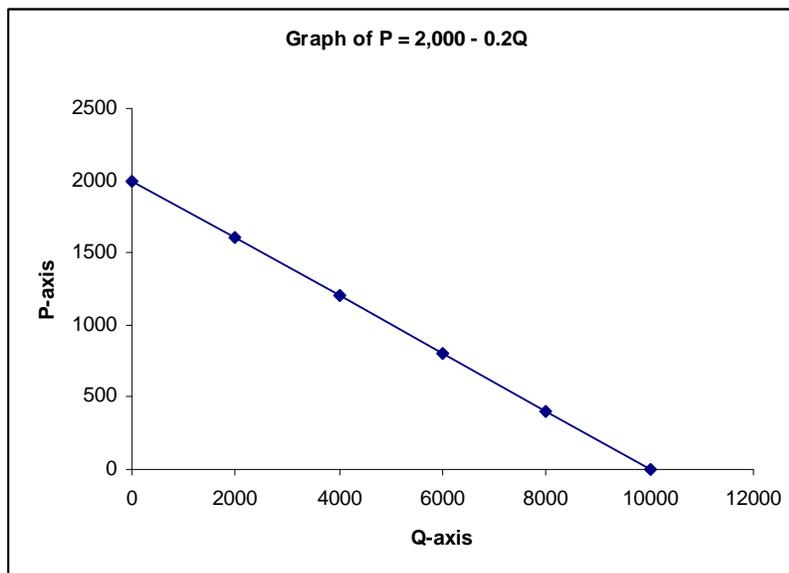


Figure 18 Graph of $P = 2,000 - 0.2Q$

You can now determine how much a bicycle should cost if 2,000 or 6,000 are demanded.

Model 2: Budget constraints

The budget constraint model describes the relationship between the prices of items, the amount of money the consumer has and the quantities of goods that can be bought with that money. For our application, we consider two types of goods, X and Y, and quantities x and y of each that will be bought.

For example, a consumer has budgeted amount W to spend on bags of vegetables (X) and bottles of water (Y).

Let the price per bag of vegetables be P_X and the price per bottle of water be P_Y . These will be known values.

Let x and y denote the (unknown) number of bags of vegetables (X) and bottles of water (Y) that can be bought with the budgeted amount W .

We can now formulate our model. The cost of buying x number of bags of vegetables is x multiplied by P_X or xP_X .

Similarly, the cost of buying y number of bottles of water is y multiplied by P_Y or yP_Y .

Since the total budget for the items is amount W , we must have $xP_X + yP_Y = W$.

The object of the model is to decide how many items we can buy — that is, to solve for x or y . The values of P_X , P_Y and W will be known.

Suppose the cost of a bag of vegetables is 200 rupees and a bottle of water costs 120 rupees. The customer has a budget of 6,250 rupees for these items.



The budget constraint model is:

$$200x + 120y = 6,250$$

We can now solve for either variable in terms of the other:

$$y = -1.67x + 52.08 \text{ or}$$

$$x = -0.6y + 31.25$$

The negative slope in both cases shows that the more we buy of one item, the less we can buy of the other.

If we buy no vegetables, we can buy about 52 bottles of water. If we buy no water, we can buy about 31 bags of vegetables. This is the meaning of the intercepts.

From the equation $y = -1.67x + 52.08$, we can see that if 20 bags of vegetables are bought, we can only buy about 18 bottles of water. (Quantities are rounded off if it is assumed that we cannot buy fractions of bags or bottles.)

Activity 1.17



Activity

Budget constraint

What will you do?

1. Draw a graph for the budget constraint problem.
2. Solve the budget constraint problem for total budget $W = 10,000$ rupees.

Model 3: Labour markets and wages

Return to the problem of balancing wages and labour. The situation is modelled by two linear equations, one from the supply side (workers) and one from the demand side (industry).

Let the number of workers available (supply) be L_s and the number of workers needed by an industry (demand) be L_d . The wages demanded by (or acceptable to) workers is W_d and the wages supplied (or offered) by the industry is W_s .

The values of the other parameters, slope and intercept, depend on the particular industry.

As an example, the two equations may be:

$$W_d = 10 + 0.5L_d \text{ and } W_s = 20 - 0.2L_s$$

Equilibrium or balance is reached when the equations are satisfied simultaneously ($W_d = W_s$) and $L_s = L_d$. Denote the labour variable by L . Then:

$$10 + 0.5L = 20 - 0.2L$$

$$0.7L = 10$$

$$L = 14.286 \text{ or } L = 14 \text{ (rounded off)}$$

The value of wages is then:

$$W = 10 + 0.5 (14) = 17$$

Activity 1.18



Activity

Check the model

What will you do?

1. Interpret the results of the labour markets and wages model determined above.
2. Determine the equilibrium point graphically.



Tip

It is very important to be able to solve and display linear equations and inequalities. These ideas are used extensively in later modules, particularly when linear programming and linear regression are discussed.



Case study: Break-even analysis

Firms that produce goods for sale have an income (revenue) from sales, but they also incur costs in the process. Generally, there are fixed costs and variable costs. Fixed costs are independent of the number of items produced, while variable costs are affected by the quantity of goods produced.

Assume that total costs equal fixed costs plus variable costs. We will also assume that variable costs are directly proportional to the number of items produced and nothing else, and that revenue comes only from selling items at a set price per unit.

Scenario: A small factory in India manufactures cell phone chargers and has determined that the price per unit will be 510.35 rupees (INR).

<i>Fixed costs:</i>	Rental	25,000
	Utilities	10,000
	Salaries	125,000
	Other	4,350

<i>Variable costs:</i>	Raw materials	0.6 INR × units produced
	Additional salaries	0.3 INR × units produced



Activity 1.19



Activity Break even report

What will you do?

Write a report on break-even for the factory that addresses these questions:

1. Denote total cost as C and number of items produced as Q . Derive the total cost equation $C(Q)$ for the factory.
2. Draw a graph of the total cost function.
3. Discuss the meaning of the values of the slope and intercept in this case.
4. Denote total income as I and derive the total income equation $I(Q)$ for the factory.
5. Graph the total income function.
6. Discuss the meaning of the values of the slope and intercept in this case.
7. Discuss and criticise the assumptions made to set up the model.
8. Define the meaning of the break-even point.
9. Calculate the break-even point algebraically and graphically.
10. Discuss your findings.



Remember these key points

- Algebra and algebraic notation are tools for analysing problems. Linear equations are the simplest type of equation that can be used to model problems. The basic form of the linear equation is $y = mx + c$, which describes the relationship between independent variable x and dependent variable y .
- The graphic representation of a linear equation is a straight line with slope m and y -intercept c in the flat plane (two-dimensional space). The graph itself consists of all points (x, y) in the plane (co-ordinate system) that satisfy the given linear relationship. The slope describes the rate of change of y with respect to x .
- Parallel lines have equal slope, but different intercepts.
- You can also consider linear inequalities of the form $y > mx + c$, or $y \leq mx + c$. In this case, you first draw the line representing the equality $y = mx + c$, then test whether $(0, 0)$ satisfies the inequality and shade in the correct region.
- Simultaneous linear equations occur when quantities have to satisfy different relationships at the same time. Solving equations simultaneously means finding the value of x and corresponding value of y that satisfy both relations. You can see this graphically when two straight lines that are not parallel intersect in a unique point (x, y) .
- If equations represent parallel lines, there will be no intersection and no simultaneous solution. If equations represent the same line, there will be infinite solutions – that is, all points on the line.

Unit summary

You have successfully completed this unit if you can:



Summary

- **identify** linear relationships between quantities;
- **draw** and **interpret** straight-line graphs;
- **solve** linear equalities and inequalities;
- **determine** solutions to simultaneous equations; and
- **apply** linear equations to management situations.

Unit 3

Non-linear functions and equations

Introduction

You have spent a lot of time on linear functions. The simple applications you saw there can be extended by using the family of non-linear functions. The simplest of these are the polynomials:

$$y(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

The a_i are given numbers and n is a natural number. We concentrate on quadratic functions where $n = 2$. The quadratic function can model situations involving accelerating objects, flight paths of projectiles, costs and profits.

Other important non-linear functions are the exponential and logarithmic functions.

Upon completion of this unit you will be able to:



Outcomes

- **solve** quadratic functions;
- **sketch** quadratic functions;
- **apply** quadratic functions in management problems; and
- **recognise** and use other non-linear functions (exponentials and logarithms).

Quadratics and their graphs

A quadratic equation has the form $y = ax^2 + bx + c$. The relationship between variables x and y is non-linear because the term x^2 appears. Remember that x^2 means $x \times x$. The values of numbers a , b and c will be given.

The shape of the graph of a quadratic is called a parabola and is symmetrical.

The parabola below describes the quadratic $y = x^2 + 2x - 3$ ($a = 1$, $b = 2$ and $c = -3$).

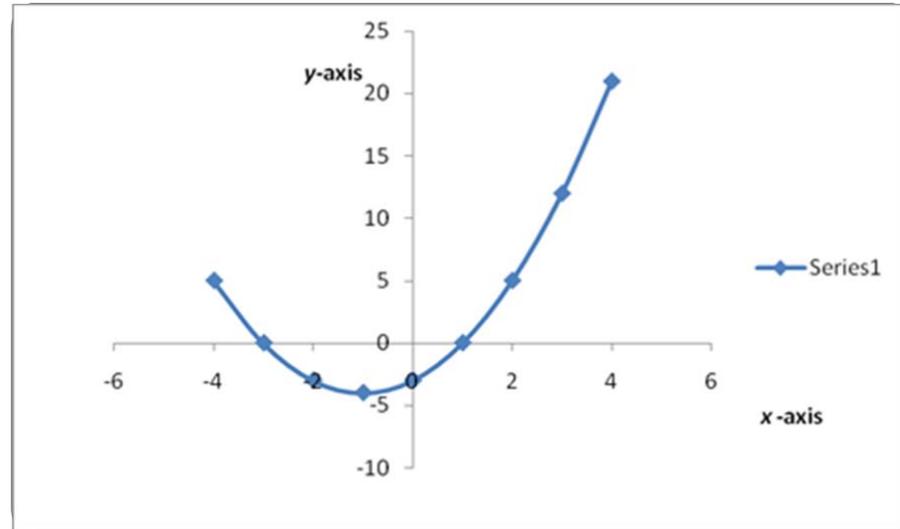


Figure 19 A parabola

The curve should be extended indefinitely on both sides.

Note that the graph can be drawn (roughly) by compiling a table of values, plotting them in the (x, y) -plane and connecting with a smooth curve.

x-values	y-values
-3	0
-2	-3
0	-3
1	0
2	5

Figure 20 Table of values for Figure 19 graph

For example, if $y(x) = x^2 + 2x - 3$, then:

$$y(-3) = (-3)^2 + 2(-3) - 3 = 9 - 6 - 3 = 0$$

For positive values of parameter a , the graph is of the basic shape \cup as above; for negative a , the parabola is turned “upside down” like this \cap . Therefore if $a > 0$ the quadratic function has a minimum and for $a < 0$ the quadratic function has a maximum.

The value of c gives the y -intercept.

There are various possibilities for the x -intercepts:

- There are no intercepts with the x -axis.
- There are two intercepts with the x -axis.

- There is one point where the graph touches the x -axis.

These possibilities are determined by the value of a crucial quantity, the determinant Δ : $\Delta = b^2 - 4ac$

These statements are true:

- If $\Delta < 0$ (negative values of Δ), then there are no intercepts with the x -axis.
- If $\Delta > 0$ (positive values of Δ), then there are two intercepts with the x -axis.
- If $\Delta = 0$, then there is exactly one point where the graph touches the x -axis.

Here are some examples:

1. Equation $y = x^2 + 2x - 3$ ($a = 1$, $b = 2$, $c = -3$)

From the equation, you see that that when $x = 0$, $y = -3$. This is the value of c , the y -intercept. From the table (Figure 20), you see that when $x = -3$ and $x = 1$, the values of y are 0. From the graph, you see that these are the points where the graph intersects the x -axis.

The value of Δ is: $\Delta = 2^2 - 4(1)(-3) = 16$, a positive number. This confirms that there should be two x -intercepts.

2. Equation $y = 2x^2 + x + 1$ ($a = 2$, $b = 1$, $c = 1$)

The value of Δ is: $\Delta = 1^2 - 4(2)(1) = -7$, a negative number. This shows that there should be no x -intercepts.

Solve equation $ax^2 + bx + c = 0$

Solving this equation means finding the values of x where $y = 0$. These are the values where the graph intersects the x -axis. A linear equation mx

$+ c = 0$ has one solution, namely $x = \frac{-c}{m}$

A quadratic equation $ax^2 + bx + c = 0$ with $\Delta > 0$ has two solutions for x , called the roots of the equation. They are given by the formulas:

$$x = \frac{-b - \sqrt{\Delta}}{2a} \text{ and } x = \frac{-b + \sqrt{\Delta}}{2a} \text{ with } \Delta = b^2 - 4ac$$

The two roots are the two x -intercepts in the graph. In this way, the formulas give an algebraic method for finding x -intercepts of parabolas.

Note that if $\Delta < 0$, then $\sqrt{\Delta}$ does not exist as a real number and there are no solutions for x in the real number system. The graph does not touch or cut the x -axis. The square root of a negative number can be calculated in the world of complex numbers, but these do not concern us here.

Therefore a quadratic always has two roots, but not necessarily roots that are real numbers.

Note that if $\Delta = 0$, then the two values of x are the same and equal to $\frac{-b}{2a}$

This value is where the graph touches the x -axis.



Draw parabolas

Earlier you learned that you can draw up a table of values, plot and connect with a smooth curve. You can also determine the x - and y -intercepts algebraically, as shown above. The value of a shows whether it is an “up” (smiling) or “down” (unsmiling) parabola. Another useful fact is that the turning point of the parabola is where $x = \frac{-b}{2a}$. This gives the maximum or minimum of quadratic function y . It is, at the same time, the point through which the axis of symmetry of the parabola runs.

Activity 1.21



Activity
Consider and draw

What will you do?

1. Consider the equation $y = -x^2 + 2x - 3$
 - a) Draw up a table of x and y values.
 - b) Find the y -intercept.
 - c) Calculate the value of Δ and interpret your answer.
 - d) Draw the graph of the quadratic.
 - e) Give the values of x and y where the quadratic reaches its maximum.
2. Consider the equation $y = x^2 + 6x + 9$
 - a) Draw up a table of x and y values.
 - b) Find the y -intercept.
 - c) Calculate the value of Δ and interpret your answer.
 - d) Will the function have a minimum or a maximum?
 - e) Determine the x -intercepts.
 - f) Give the coordinates (x, y) of the minimum point.
 - g) Draw the graph of the quadratic.

Here's our feedback

1. a)

x-values	y-values
-3	-18
-2	-11
-1	-6
0	-3
1	-2
2	-3
3	-6

Figure 21 X and y values for $y = x^2 + 6x + 9$

b) If $x = 0$, then $y = 3$.

c) $\Delta = 22 - 4(-1)(-3) = 4 - 12 = -8$. The determinant is negative, so there are no x-intercepts (no real solutions to the equation $-x^2 + 2x - 3 = 0$).

d) $a = -1$ implies that the shape is \cap .

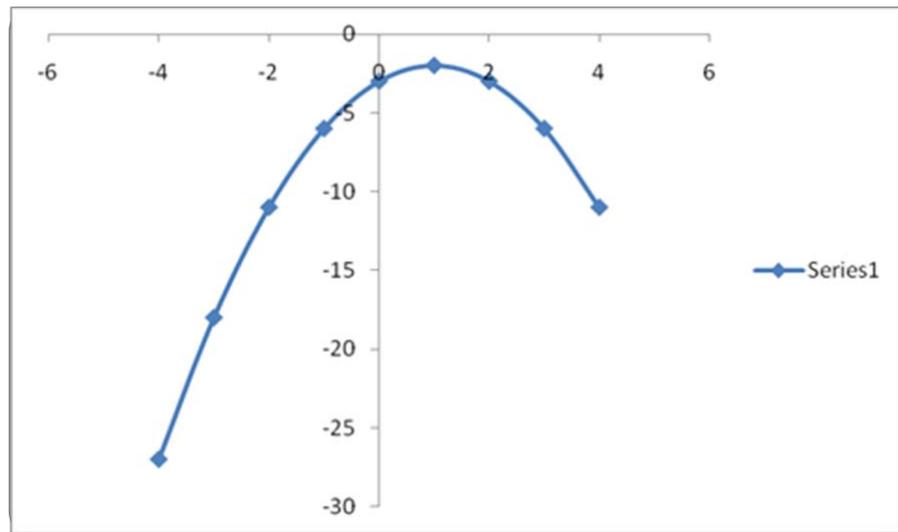


Figure 22 The quadratic for $y = x^2 + 6x + 9$

e) There will be a maximum at $x = -\frac{2}{2(-1)} = 1$. The corresponding value for y is $y = -(1)^2 + 2(1) - 3 = -2$.



2. a)

x-values	y-values
-4	1
-3	0
0	9
1	16
2	25

Figure 23 X and y values for $y = x^2 + 6x + 9$

b) $y = 9$

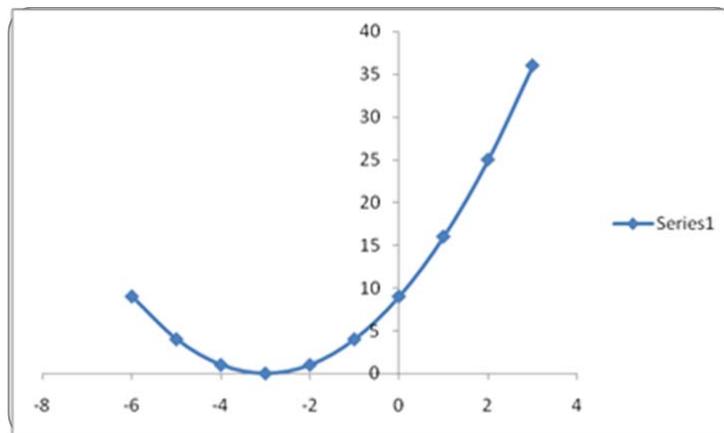
c) $\Delta = 6^2 - 4(1)(9) = 36 - 36 = 0$. The graph should touch the x-axis in one point.

d) Since we have $a > 0$ ($1 > 0$), the basic shape is \cup and the function has a minimum.

e) $x = \frac{-b - \sqrt{\Delta}}{2a}$ and $x = \frac{-b + \sqrt{\Delta}}{2a}$ where $\Delta = 0$ in this case.

Therefore the solution is $x = \frac{-6}{2} = -3$.

f) $x = -3$ and $y = (-3)^2 + 6(-3) + 9 = 0$. Minimum point is $(-3, 0)$.

Figure 24 Quadratic for $y = x^2 + 6x + 9$

Activity 1.22



Activity

Draw some more

What will you do?

1. Consider the equation $y = 2x^2 + 3x - 2$
 - a) Draw up a table of x and y values.
 - b) Find the y -intercept.
 - c) Calculate the value of Δ and interpret your answer.
 - d) Draw the graph of the quadratic.
 - e) Give the values of x and y where the quadratic reaches its maximum.

Exponentials, logarithms and their graphs

Not all functions are polynomials. The class of all functions is enormous, but we will consider only exponential and logarithmic functions here.

Exponential growth means that something is increasing very quickly. The general form of the exponential function is: $y = K a^x$, where a can be any positive number, K any number (positive for growth and negative for decay) and x can take on positive or negative values. Exponential functions take on positive values only — the graphs do not intercept the x -axis. As x increases, the graph shoots up for $K > 0$ to positive infinity.

Here are some examples:

1. $y = 2^x$ represents a situation where y doubles in value for each successive positive whole value of x . (Note that 2^x is often expressed as $2^{\wedge}x$ in computer language.)

x-values	y-values
0	1
1	2
2	4
3	8
4	16

Figure 25 X and y values for $y = 2^x$

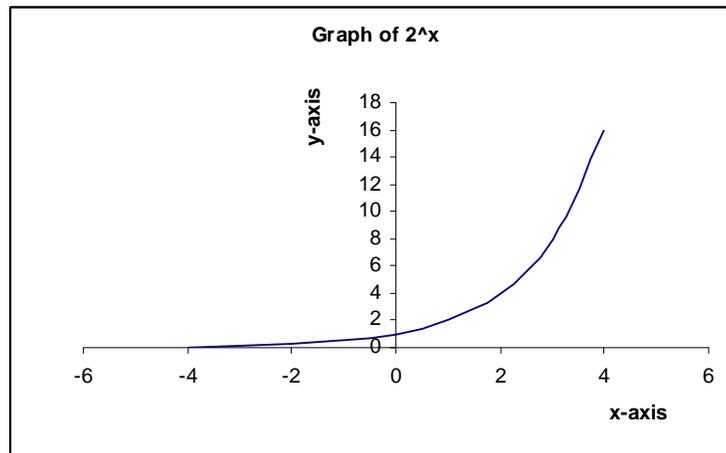


Figure 26 Graph of $y = 2^x$

- The expression $A = P(1 + r)^n$ describes the growth of amount P with compound interest r (expressed as a decimal) over n time periods. You encountered this function in Unit 1. It is an exponential $y = Ka^x$ with $K = P$, $a = (1 + r)$ and $x = n$.
- Another well-known and useful exponential function is $y = e^x$. The number e ($e \approx 2.7183$) is the special irrational number you met in Unit 1. Scientific calculators have a key for the e function, which is important in describing population growth and continuous compounding. Continuous compounding in finance means that the compounding takes place every instant. (It is also used in Poisson distribution, covered in Module 2.)

The graph of e^x looks very similar to that of 2^x in Figure 26, but with a steeper slope.

Properties

- $a^{x+z} = a^x a^z$
- $a^{x-z} = \frac{a^x}{a^z}$

Logarithms

Logarithms are the inverse functions to exponentials. We will only look at the natural logarithm, the \log_e or \ln -function. It is the inverse of the e -function, meaning that if we apply one function after the other, they cancel each other out:

$\ln(e^x) = x$ for any x and $e^{\ln x} = x$ for any positive value of x

This shows that $\ln(e^0) = 0$. That is, $\ln 1 = 0$. The equation $\ln x = 0$ therefore has solution or root $x = 1$. The graph intercepts the x -axis at $x = 1$. We also note that the \ln -function $\ln(x)$ is only defined for positive x .

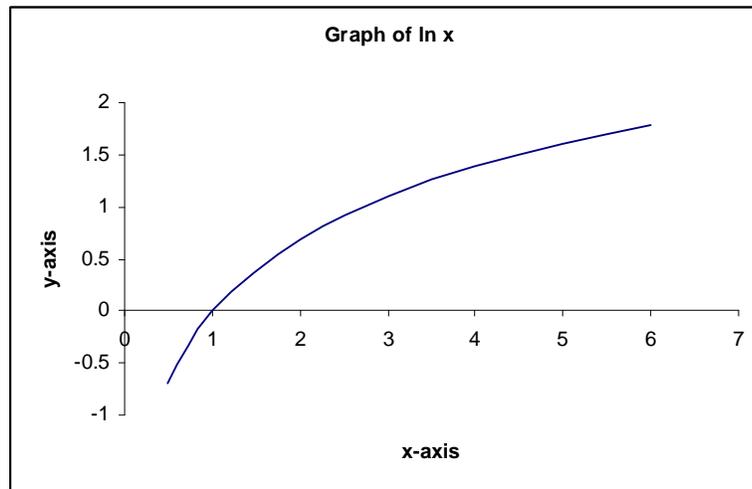


Figure 27 Graph of $\ln(e^x)$

As x increases, the graph rises, but very slowly. As x tends to 0, the graph falls rapidly towards negative infinity.

Properties

- $\ln(xz) = \ln x + \ln z$
- $\ln\left(\frac{x}{z}\right) = \ln x - \ln z$
- $\ln(A^x) = x \ln A$

Logarithms are important when you want to solve problems like the ones you encountered with annuities in Unit 1:

Find n in terms of given A , P and r if $A = P(1+r)^n$

First, separate terms containing the unknown n from the other terms:

$$(1+r)^n = \frac{A}{P}$$

Then apply the \ln -function on both sides:

$$\ln(1+r)^n = \ln \frac{A}{P}$$

Now use properties 3 and 2 above:

$$n \ln(1+r) = \ln A - \ln P$$

Now solve for n :

$$n = \frac{\ln A - \ln P}{\ln(1+r)}$$



Activity 1.23



Activity Solve logarithms

What will you do?

- Use a pocket calculator to determine the values of:
 - e^3
 - e^4
 - e^7 and compare this to $e^3 e^4$
- Complete the table and discuss.

x -values	x^3 -values	3^x -values
0		
1		
2		
3		
4		

- Solve for n if:
 - $(1.1)^n = 40$
 - $20(1 + 0.05)^n = 10$

Apply non-linear functions

Quadratic equations applied to supply and demand functions are discussed here, but in Modules 4 and 5 we will look at more advanced topics in risk management where non-linear functions are used.

The supply and demand for a particular product are described by these functions:

$$P_S = Q_S^2 + 2Q_S + 16 \text{ and } P_D = Q_D^2 - 2Q_D + 20$$

P_S is the price per item that the producer will accept (supply price) and P_D is the price the buyer is willing to pay. Q_S is the number of products supplied and Q_D the number of products demanded. The units of Q are in thousands and the units of P in tens of dollars. Market equilibrium occurs when supply equals demand: $Q_S = Q_D$ and $P_S = P_D$. The variables are then called Q and P without subscripts.

We want to display the situation graphically and find the equilibrium price and quantity for the situation.

1. Draw the graphs:

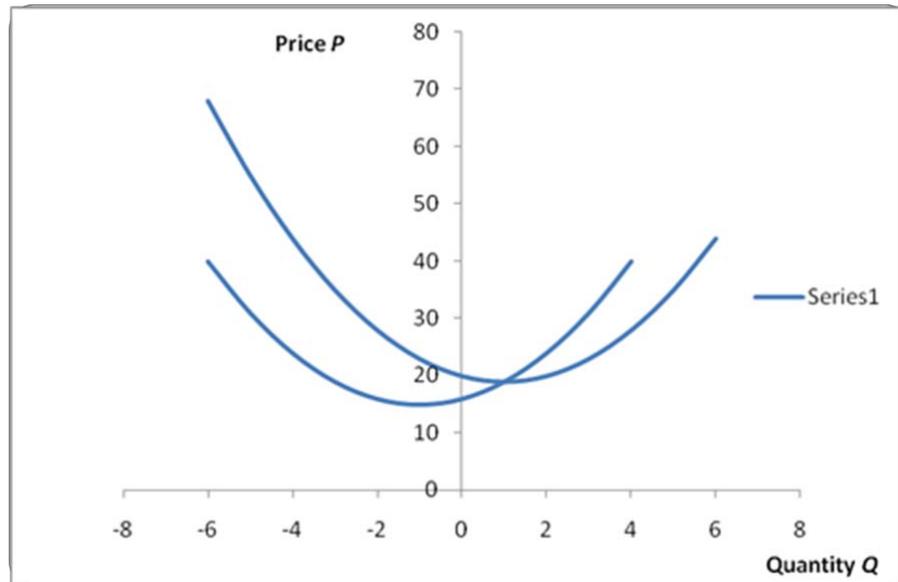


Figure 28 Graphs for $P_S = Q_S^2 + 2Q_S + 16$ and $P_D = Q_D^2 - 2Q_D + 20$

2. Equilibrium occurs when both quadratics are satisfied. From the graph, it appears that the equations are satisfied simultaneously at $Q = 1, P = 19$. This means $Q = 1,000$ items and $P = 190$ dollars (approximately).
3. Algebraically, we have (from the fact that $P_S = P_D$ and $Q_S = Q_D$):
 $Q^2 + 2Q + 16 = Q^2 - 2Q + 20$. This gives $4Q = 4$ or $Q = 1$. This represents 1,000 items to be manufactured. The corresponding value for P is:

$$P = Q^2 + 2Q + 16 = 1 + 2 + 16 \text{ (at } Q = 1) = 19$$

This represents a price per item of 190 dollars.



**Case study:
Supply and demand**

The supply and demand for a product manufactured in your factory is described by the following functions:

$$P_S = 2Q_S^2 + Q_S - 4 \text{ and } P_D = Q_D^2 - 2Q_D + 3$$

Quantities are in hundreds and prices in thousands of units of your country's currency.

Separately for each function, determine the minimum price and quantities acceptable to buyers and to yourself.

Then determine the market equilibrium and discuss your findings.

Remember these key points

- Quadratic equations have the form $y = ax^2 + bx + c$.
- The shape of the graph is symmetrical and called a **parabola**.
- For positive values of a , the graph has the basic shape \cup and for negative a , the parabola is turned “upside down” as \cap . The value of c gives the y -intercept. The quadratic has either a minimum or maximum at the point where $x = \frac{-b}{2a}$ and the graph is symmetrical about the vertical line through this point.
- A quadratic equation with $\Delta > 0$ has two solutions for x :
$$x = \frac{-b - \sqrt{\Delta}}{2a} \text{ and } x = \frac{-b + \sqrt{\Delta}}{2a} \text{ where } \Delta = b^2 - 4ac$$
- If $\Delta = 0$, then the two values of x are the same and equal to $\frac{-b}{2a}$
This value is where the graph touches the x -axis. If $\Delta < 0$, then $\sqrt{\Delta}$ does not exist (as a real number) and there are no real solutions for x . The graph does not touch or cut the x -axis. However, there will be two complex numbers as roots.
- Polynomial equations of degree n have the form $y = a_1x^n + a_2x^{n-1} + \dots + a_nx + a_{n+1}$ and have n roots.
- The exponential function is $y = K a^x$ where a can be any positive number, K any number and x can take positive or negative values. Exponential functions take on **positive values only**: the graphs do not intercept the x -axis. Of particular importance is the function $y = e^x$, with its inverse the natural logarithm $\ln x$.



Unit summary

You have successfully completed this unit if you can:



Summary

- **understand** quadratic relationships between quantities;
- **draw** and **interpret** parabolas;
- **solve**, or **find the roots of**, quadratic equations;
- **apply** quadratic equations in management problems; and
- **recognise** and **use** exponential and logarithmic functions.

Unit 4

Differentiation and integration

Introduction

Calculus is the branch of mathematics that studies changes in quantities. Rate of change is described by the process of differentiation, and integration is the “reverse” process of differentiation. Integration can also be seen as a process of summing and averaging.

Upon completion of this unit you will be able to:



Outcomes

- **differentiate** simple functions;
- **calculate** rates of change of quantities;
- **perform** basic integration;
- **determine** maxima and minima of quantities; and
- **apply** differentiation and integration in management problems.

Differentiation

Consider again a linear function such as $y = 5x + 3$ and its straight-line graph. The slope is $m = 5$ and you saw that this means the following: if x changes by one unit in the relation, then y changes by five units. The **rate of change** of y with respect to x is 5 and this remains constant for all values of x . In the language of calculus, we say the derivative of y with respect to x is 5 and we write $\frac{dy}{dx} = 5$ or $\frac{d}{dx}y = 5$. Some books use the notation $y'(x) = 5$.

Derivatives of polynomials

For any relationship $y = mx + c$, we have that $\frac{dy}{dx} = m$.

Formally, the operator $\frac{d}{dx}$ means rate of change with respect to x of any quantity (for example, y) to which it is applied, and is called the derivative with respect to x . Graphically, it gives the slope.

For any constant quantity c , we have $\frac{dc}{dx} = 0$. This is because the graph of a constant is a horizontal line, which of course has a slope of zero (0).

Now for some non-rigorous reasoning:



If $y = x$, then we have

$$\frac{dy}{dx} = \frac{d}{dx}x \text{ (apply operator } \frac{d}{dx} \text{ right through).}$$

But the slope of $y = x$ equals 1, and so

$$1 = \frac{d}{dx}x$$

Therefore we have the following rules:

- $\frac{dc}{dx} = 0$ for any constant c
- $\frac{d}{dx}x = 1$
- $\frac{d}{dx}mx = m$ $\frac{d}{dx}x = m$
- $\frac{dy}{dx} = m$ for $y = mx + c$

Now consider quadratic $y = x^2 + 6x + 9$ and its parabolic representation.

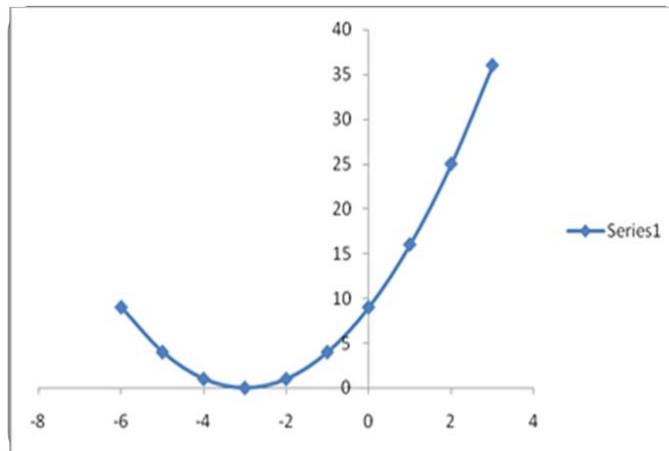


Figure 29 Graphic of $y = x^2 + 6x + 9$

What is the rate of change of y with respect to x ? In other words, what is the slope of the graph? Clearly the slope (steepness) is not constant and changes as x changes: the slope at $x = -1$ is less than the slope at $x = 2$.

Measure the slope at a point (x, y) on the graph by drawing a straight line at the point that just touches the graph there. Such a line is called a tangent line. Draw the tangents at points (x, y) where $x = -1$ and $x = 2$, respectively.

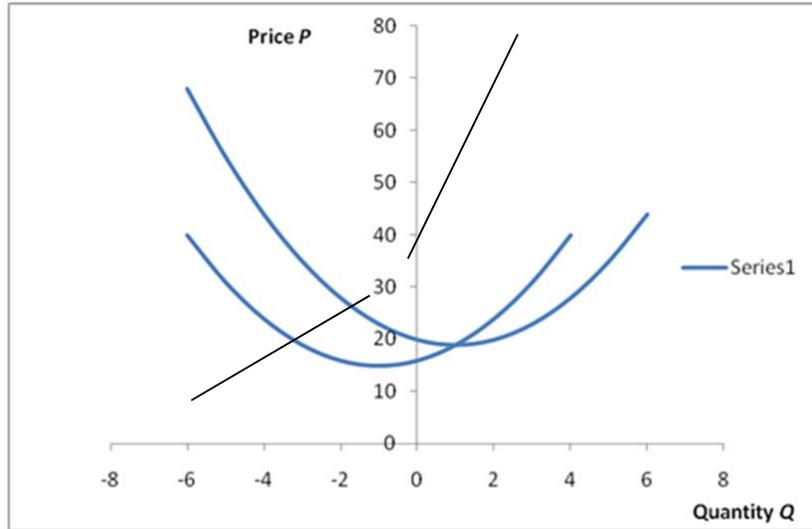


Figure 30 Graphic showing tangents at points (x, y) where $x = -1$ and $x = 2$

The slope of the tangent at $x = -1$ is clearly less than at $x = 2$. From the graph, you can calculate the slopes: at $x = -1$, you get a slope of 4 and at $x = 2$, you get a slope of 10. **The rate of change** of y with respect to x changes for different values of x . How do you calculate this rate of

change $\frac{dy}{dx}$ without having to draw graphs? Using the language of calculus, the derivative of $y = x^2 + 6x + 9$ with respect to x is:

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} [x^2 + 6x + 9] \\ &= \frac{d}{dx} x^2 + \frac{d}{dx} 6x + \frac{d}{dx} 9 \\ &= \frac{d}{dx} x^2 + 6 + 0 \end{aligned}$$

We now give a rule without proof: $\frac{d}{dx} x^2 = 2x$

Therefore $\frac{d}{dx} [x^2 + 6x + 9] = 2x + 6$

The slope or rate of change of $y = x^2 + 6x + 9$ is therefore:

$$\frac{d}{dx} [x^2 + 6x + 9] = 2x + 6$$

Clearly, the slope changes as x changes. At $x = -1$, you get a slope of 4 and at $x = 2$, you get a slope of 10. At $x = -3$, you get a slope of 0 because the tangent there is a horizontal line.



Remember these rules

- $\frac{d}{dx} x^2 = 2x$
- $\frac{d}{dx} [ax^2 + bx + c] = 2ax + b$
- $\frac{d}{dx} ax^n = a nx^{n-1}$

Activity 1.25



Activity
Find the derivative

What will you do?

1. Find the derivative $\frac{dy}{dx}$ if:
 - a) $y = 2x + 4$
 - b) $y = -5x + 20$
 - c) $y = 3x^2 + x + 3$
 - d) $y = -0.5x^2 - 6x + 2$
 - e) $1.5y = 5x^4 - 6x^3 + 2x$
2. Quantities P and Q are related by the equation $P = 2Q^2 - 3Q + 1$. If $Q = 10$, what is the value of P ? What is the rate of change of P with respect to Q at this point?

Here's our feedback

1. a) $\frac{dy}{dx} = 2$
 - b) $\frac{dy}{dx} = -5$
 - c) $\frac{dy}{dx} = 6x + 1$
 - d) $\frac{dy}{dx} = -0.5(2)x - 6 = -x - 6$
 - e) $\frac{dy}{dx} = 20x^3 - 18x^2 + 2$
2. If $Q = 10$, $P = 171$. $\frac{dP}{dQ} = 4Q - 3$. At $Q = 10$, $\frac{dP}{dQ} = 37$
-

Activity 1.26



Activity

Solve derivatives

What will you do?

- Find the derivatives $\frac{dy}{dx}$ if:
 - $y = 5x + 6$
 - $y = -0.3x + 2$
 - $y = 2x^2 + 4x + 3$
 - $y = -0.8x^2 - x + 12$
 - $y = 6x^3 - 0.5x^2 + 3x + 4$
- Explain the meaning of the equation $\frac{dy}{dx} = -3$
What relationship can be expected between y and x ?
- Explain the meaning of the equation $\frac{dy}{dx} = 2x + 4$
What relationship can be expected between y and x ? If the relationship is expressed graphically, what is the rate of change of y with respect to x at $x = 4$?
- Quantities P and Q are related by the equation $2P = 6Q^2 - Q + 4$
If $Q = 1$, what is the value of P ? What is the rate of change of P with respect to Q at this point?
- What is the slope of the line describing the relation $5x - 3y - 12 = 0$?

Other important derivatives and rules

Calculus is a huge subject and we cannot cover all aspects. Rules dealing with the differentiation of special functions, products of functions or quotients of functions can be found in textbooks on calculus or on the Internet.

Here are two other useful derivatives:

- If $y = e^x$, then $\frac{dy}{dx} = e^x$, and if $y = e^{kx}$ for some constant k , then $\frac{dy}{dx} = ke^{kx}$
- If $y = \ln x$, then $\frac{dy}{dx} = \frac{1}{x}$

If you differentiate the function y twice, you denote this as:

$$\frac{d}{dx} \left[\frac{dy}{dx} \right] \text{ or } \frac{d^2y}{dx^2}$$



For example, if $y = 3x^2 + 2x$ then $\frac{d^2y}{dx^2} = \frac{d}{dx} \left[\frac{dy}{dx} \right] =$

$$\frac{d}{dx} [6x + 2] = 6$$

Why do you need to know about derivatives? You have already seen that differentiation gives the rate of change of quantities, which is a very important concept. Another important application of differentiation is that it enables us to find the maxima or minima of complicated functions.

Optimisation: Maxima and minima

One of the most important applications of derivatives in management is to find **optimal values of quantities**. You can also discover where a quantity is increasing or decreasing.

Optimisation of linear functions

For a linear relationship $y = mx + c$, the slope m tells us whether the function is increasing or decreasing. If $m > 0$, the function is increasing and if $m < 0$, the function is decreasing. The straight-line graph extends indefinitely and therefore there are no maxima or minima **unless the function is restricted to certain values of x** .

Suppose $m > 0$ and x is restricted to the domain $a \leq x \leq b$. This means y is increasing from $x = a$ to $x = b$. The minimum of y will be at $x = a$ and the maximum of y will be at $x = b$. The actual minimum value of y will be $y = ma + c$ and the maximum value of y will be $y = mb + c$.

Suppose $m < 0$ and x is again restricted to the domain $a \leq x \leq b$. This means y is decreasing from $x = a$ to $x = b$. The minimum of y will be at $x = b$ and the maximum of y will be at $x = a$. The actual minimum value of y will be $y = mb + c$ and the maximum value of y will be $y = ma + c$.

Optimisation of quadratics

The parabola that describes the relationship $y = ax^2 + bx + c$ increases and decreases. The shape is \cup for $a > 0$ and shows that the function first decreases and then increases as x increases from left to right. Function y has a minimum at the turning point.

The shape is \cap for $a < 0$ and shows that the function first increases and then decreases as x increases from left to right. Function y has a maximum at the turning point.

In the previous unit, this turning point was stated as being at $x = \frac{-b}{2a}$

How did we know that? Note that when the graph of a function has a turning point, the slope of the graph becomes zero at that point. This means:

$$\frac{dy}{dx} = 0 \text{ or } 2ax + b = 0 \text{ at the turning point. Solve for } x:$$

$$2ax = -b$$

$$x = \frac{-b}{2a}$$

Substituting this value into the equation for y yields the optimal value of y .

General rule

Maxima and minima for more general functions are obtained in the same way, by setting $\frac{dy}{dx} = 0$ and solving for x . This works because maxima

and minima lie at the turning points. The test for a minimum is that $\frac{d^2y}{dx^2}$

> 0 and the test for a maximum is that $\frac{d^2y}{dx^2} < 0$ at that point x .



Note it!

Maxima and minima for higher-degree polynomials

A quadratic has either a **minimum** or a **maximum**. A polynomial of degree 3 can have both a maximum and a minimum.

Example 1: Consider $y = x^3 - 6x^2 + 11x - 6$

$$\frac{dy}{dx} = 3x^2 - 12x + 11$$

$\frac{dy}{dx} = 0$ implies $x = 2.577$ and $x = 1.423$ are points where y reaches optima.

$\frac{d^2y}{dx^2} = 6x - 12 = 3.463 (> 0)$ at $x = 2.577$ and $\frac{d^2y}{dx^2} = -3.462 (< 0)$ at $x = 1.423$.

Therefore we have a minimum at $x = 2.577$ and a maximum at $x = 1.423$. See the graph below (Figure 31).

Example 2: Consider $y = x^3$

$$\frac{dy}{dx} = 0 \text{ implies } 3x^2 = 0 \text{ and } x = 0 \text{ and } x = 0 \text{ are the two roots.}$$

Do we have a maximum or minimum at $x = 0$?

Since $\frac{d^2y}{dx^2} = 6x = 0$ when $x = 0$, we have neither. In fact, there is a point of inflection. See the second graph (Figure 32) below.

Polynomials of degrees higher than 3 can have more than one local maximum/minimum, as well as points of inflection. It is important to check carefully whether you have found the global maximum/minimum.

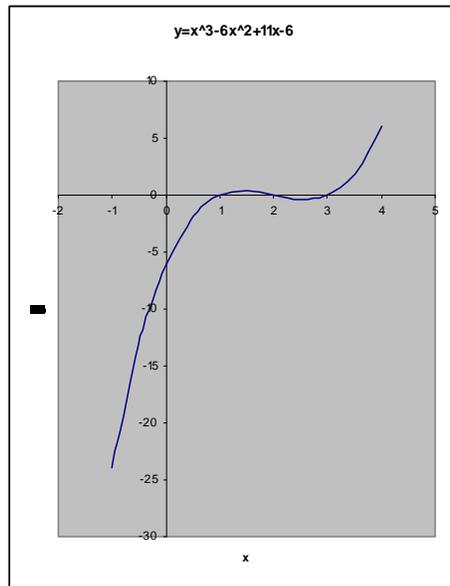


Figure 31 Graph of $y = x^3 - 6x^2 + 11x - 6$

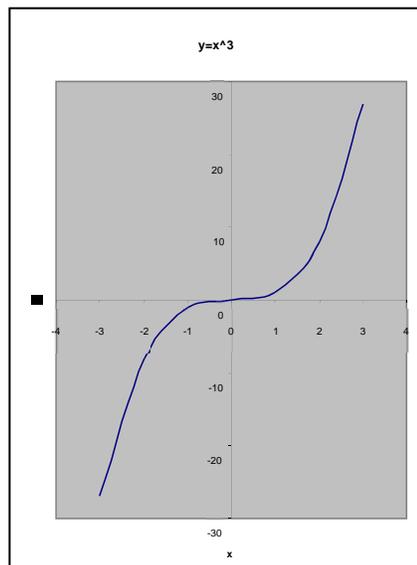


Figure 32 Graph of $y = x^3$

Activity 1.27



Activity Use calculus methods

What will you do?

Use the methods of calculus to answer these questions:

1. Consider function $y = -2x + 6$ with constraint $x \geq 1$.
 - a) Is the function increasing or decreasing? Justify your answer.
 - b) Find the minima and maxima (if they exist).
 - c) Calculate $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

2. Consider function $y = 3x^2 + 6x - 4$
 - a) Calculate $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$
 - b) Is the function increasing or decreasing? Justify your answer.
 - c) Find the minima and maxima (if they exist).
 - d) If you add the constraint $-2 \leq x \leq 2$, how does this change your answer for c)?

Here's our feedback

1. Consider function $y = -2x + 6$ with constraint $x \geq 1$.
 - a) The function is decreasing because the slope is negative
 $\left(\frac{dy}{dx} = -2\right)$
 - b) The function has a maximum at $x = 1$. The value of the maximum is $y = -2(1) + 6 = 4$. The function has no minimum.
 - c) $\frac{dy}{dx} = -2$ and $\frac{d^2y}{dx^2} = \frac{d}{dx}[-2] = 0$

2. Consider function $y = 3x^2 + 6x - 4$.
 - a) $\frac{dy}{dx} = 6x + 6$ and $\frac{d^2y}{dx^2} = \frac{dy}{dx} [6x + 6] = 6$
 - b) The function is decreasing for $\frac{dy}{dx} < 0$.
 This is for $6x + 6 < 0$; that is, for $x < -1$

 The function is increasing for $\frac{dy}{dx} > 0$.
 This is for $6x + 6 > 0$; that is, for $x > -1$



c) The turning point is where $\frac{dy}{dx} = 0$; that is, where $x = -1$. Since

$\frac{d^2y}{dx^2} = 6$ and $6 > 0$, the function has a minimum at $x = -1$. The minimum value is $y = 3(-1)^2 + 6(-1) - 4 = -7$. There is no maximum. Recall that the basic shape is \cup .

d) You can now expect a maximum for x in the domain $-2 \leq x \leq 1$. The maximum will lie at one of the extremities: $x = -2$ or $x = 1$. The corresponding y -values are $y = 3(-2)^2 + 6(-2) - 4 = -4$ or $y = 3(1)^2 + 6(1) - 4 = 5$.

The maximum value is $y = 5$ at $x = 1$.

Activity 1.28



Activity
Use more calculus
methods

What will you do?

Use the methods of calculus to answer these questions:

- Consider function $y = 4x - 2$ with constraint $x \geq -1$.
 - Is the function increasing or decreasing? Justify your answer.
 - Find the minima and maxima (if they exist).
 - Calculate $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$
- Consider function $y = -x^2 + 2x - 3$
 - Calculate $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$
 - Is the function increasing or decreasing? Justify your answer.
 - Find the minima and maxima (if they exist).
 - If you add the constraint $-2 \leq x \leq 1$, how does this change your answer for 2c)?
- A certain function T has a turning point at $s = 6$.
The value of $\frac{dT}{ds}$ is $\frac{dT}{ds} = s^2 - 36$
Does the function have a minimum or maximum at $s = 6$?
- If $\frac{dP}{dQ} = -2x + 5$, can you guess what type of relationship may exist between P and Q ?

Integration

At a simplified level, integration can be seen as the reverse operation to differentiation. For example, suppose you are given the rate of change of

Profit P with respect to Quantity Q , namely $\frac{dP}{dQ}$, as $\frac{dP}{dQ} = -4Q + 77$

Can you find the original expression for P as a function of Q ?

Indefinite integrals

You can guess that the original expression contains Q^2 and also $77Q$, because differentiating these terms will leave you with factor $2Q$ and the constant 77 . But you need $-4Q$ after differentiating. So you need $-2Q^2$ in

the original expression. A good guess is that if $\frac{dP}{dQ} = -4Q + 77$, then

$P(Q) = -2Q^2 + 77Q$. (Check this.)

But if $P = -2Q^2 + 77Q + 1$, you also get $\frac{dP}{dQ} = -4Q + 77$ since the derivative (slope) of constant 1 is zero.

In general, we say if $\frac{dP}{dQ} = -4Q + 77$, then $P(Q) = -2Q^2 + 77Q + K$, where K can be any constant number.

Mathematically, we write: $\int \frac{dP}{dQ} dQ = P$

The integral \int reverses the derivative of P and gives us back P . In this specific case:

$$\int \frac{dP}{dQ} dQ = P(Q) = \int (-4Q + 77) dQ = -2Q^2 + 77Q + K$$

where K can be any constant number.

We say the integral of $(-4Q + 77)$ is $(-2Q^2 + 77Q + K)$, meaning that if we differentiate the latter expression we get back the first.

Is there any way to get the value of K ? We need both $\frac{dP}{dQ}$ and one specific value for P – usually the value of P when $Q = 0$.

If we are given $\frac{dP}{dQ} = -4Q + 77$ and the data that if $Q = 0$, then $P = -150$, we can find P exactly.

From $P(Q) = -2Q^2 + 77Q + K$ and setting $Q = 0$, we have $P(0) = K$. But we know $P(0) = -150$ in this case, so that we must have $K = -150$. Therefore $P = -2Q^2 + 77Q - 150$.



In general:

$$\int \frac{dy}{dx} dx = y(x) + K \text{ or}$$

if f is a function of x , then $\int f(x)dx$ is the new function that, after differentiation, gives back f .

If we integrate powers of x , the new expression must contain powers of x of one degree higher. If we integrate constants, the new expression must contain variable x . If we integrate zero, the new expression is a constant. We need data to find the exact value of this constant.

Note that the integrals considered here are all **indefinite integrals**.

Activity 1.29



Activity

Let's practice!

What will you do?

1. Calculate the indefinite integrals:

a) $\int 2dx = 2x + K$

b) $\int 3dx = \frac{3}{2}x^2 + K$

c) $\int (1 - 2x)dx = x - x^2 + K$

d) $\int (3x^2 + 8)dx = x^3 + 8x + K$

2. Find $Z(T)$ if $\frac{dZ}{dT} = -T^2 + T$ and $Z(0) = 4$

Here's our feedback

1. a) Test: $\frac{d}{dx} [2x + K] = 2 + 0 = 2$

b) Test: $\frac{d}{dx} [\frac{3}{2}x^2 + K] = 2(\frac{3}{2})x + 0 = 3x$

c) Test: $\frac{d}{dx} [x - x^2 + K] = 1 - 2x + 0$

d) Test: $\frac{d}{dx} [x^3 + 8x + K] = 3x^2 + 8$

$$2. Z(T) = \int \frac{dZ}{dT} dT = \int (-T^2 + T) dT = -\frac{1}{3}T^3 + \frac{1}{2}T^2 + K.$$

$$Z(0) = 4 = K, \text{ so } Z(T) = -\frac{1}{3}T^3 + \frac{1}{2}T^2 + 4$$

$$\text{Test: } \frac{d}{dT} [-\frac{1}{3}T^3 + \frac{1}{2}T^2 + 4] = 3(-\frac{1}{3})T^2 + 2(\frac{1}{2})T + 0 = -T^2 + T$$

Activity 1.30



Activity

Calculate the integrals

What will you do?

- $\int (2x - 2) dx$
- $\int (-x + 1) dx$
- $\int (4 - 4x) dx$
- $\int (x^2 + 3x - 1) dx$
- Find $P(Q)$ if $\frac{dP}{dQ} = 5$ and it is given that the graph has P -intercept equal to 4. Draw the graph of P .
- Find $S(t)$ if $\frac{dS}{dt} = -2t + 2$ and it is given that the graph has S -intercept equal to 10. Draw the graph and determine the maximum value of S .

Definite integrals

Definite integrals provide **bounds** for the process of integration.

They have the form $\int_a^b y dx$. The numbers a and b are the bounds for the variable x . What this means is that when you have the integrated expression as a function of x , you substitute the upper bound b and then deduct the expression where you have substituted the lower bound a .

So $\int (2x - 1) dx = x^2 - x + K$ and

$$\begin{aligned} \int_1^4 (2x - 1) dx &= [x^2 - x + K](\text{for } x = 4) - [x^2 - x + K](\text{for } x = 1) \\ &= [4^2 - 4 + K] - [1^2 - 1 + K] = 16 - 4 - 0 = 12 \end{aligned}$$

Notice that K has vanished.

Here are some more examples:

- $\int_0^{10} (4x + 3) dx = [2x^2 + 3x + K](\text{for } x = 10) - [2x^2 + 3x + K](\text{for } x = 0)$
 $= [2(10)^2 + 3(10) + K] - [2(0)^2 + 3(0) + K] = 230 - 0 = 230$



$$\begin{aligned}
 2. \int_{-1}^1 x^2 dx &= \left[\frac{1}{3}x^3 + K \right] (\text{for } x=1) - \left[\frac{1}{3}x^3 + K \right] (\text{for } x=-1) \\
 &= \frac{1}{3}(1)^3 - \frac{1}{3}(-1)^3 \\
 &= \frac{1}{3} + \frac{1}{3} = \frac{2}{3}
 \end{aligned}$$

Definite integrals as areas

One interpretation of integration is that it gives the net area under a graph, above the x -axis.

Consider $\int_0^{10} (2x+3)dx$ and draw the graph of $y=2x+3$, $0 \leq x \leq 10$

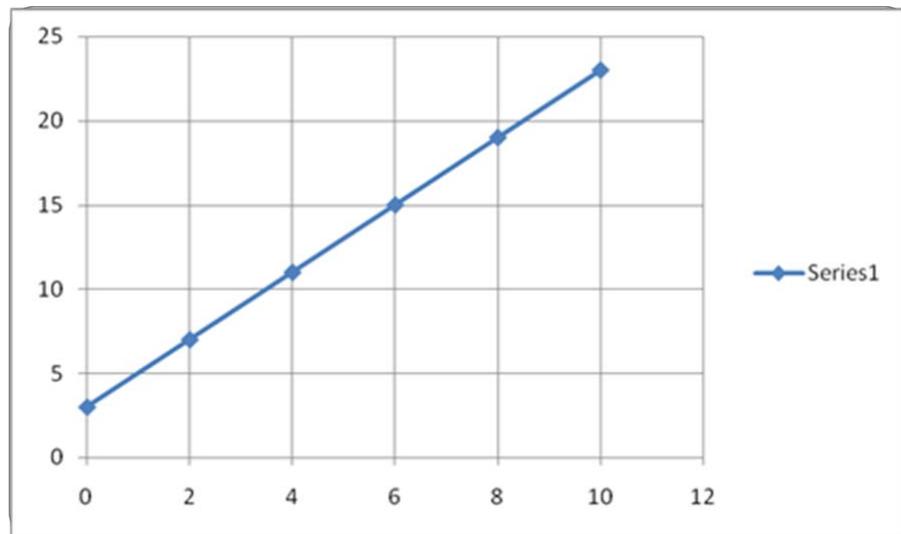


Figure 33 Graph of $y=2x+3$, $0 \leq x \leq 10$

The area under the graph, above the horizontal axis and within the bounds of x , is the area of a right-angled triangle with base 10 units and height 20 units *plus* the area of a (10×3) rectangle.

Area of triangle = $\frac{1}{2}(\text{base} \times \text{perpendicular height}) = \frac{1}{2}(200) = 100$ square units. Area of rectangle is $10 \times 3 = 30$ square units. The total area under the graph is 130 square units.

Now calculate that $\int_0^{10} (2x+3)dx = 130$

Note: If the definite integral gives a negative answer, it means the area lies *below* the x -axis.

Activity 1.31



Activity

What will you do?

Calculate $\int_{-1}^1 x dx$ and interpret your answer.

Applications of differentiation and integration

You are now ready to apply your techniques to real-life problems.

Break-even and maximum profit

A farming co-op produces rice, which is sold in 50 kilogram bags. Let Q denote the number of bags produced per week during harvest time. Costs C are denoted in units of dollars.

The total weekly cost function for producing the product is $C = 3Q + 150$

The total weekly income function for this product is $I = 80Q - 2Q^2$

You have been assigned to analyse the situation for break-even and profit.

1. Sketch the two functions.

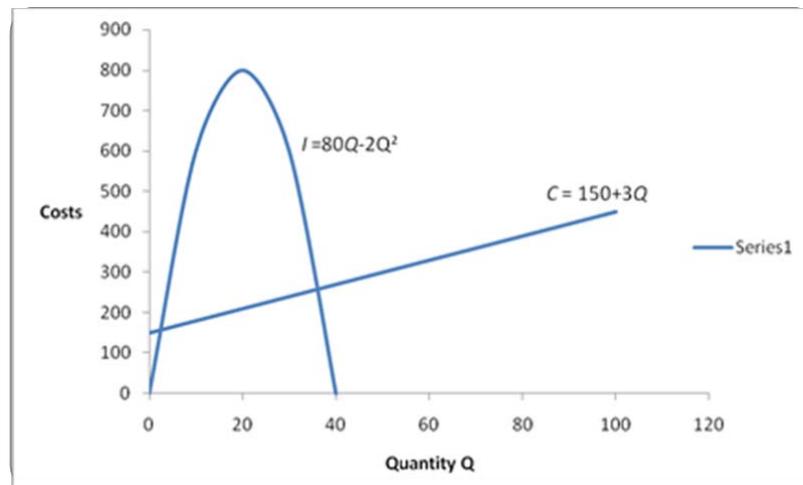


Figure 34 Weekly cost and income functions

The graph in Figure 34 shows both Income and Costs.

2. Determine the profit function P as a function of Q and sketch it.

Profit = Income – Costs.

$$\text{Therefore } P = 80Q - 2Q^2 - (3Q + 150)$$

$$= -2Q^2 + 77Q - 150$$

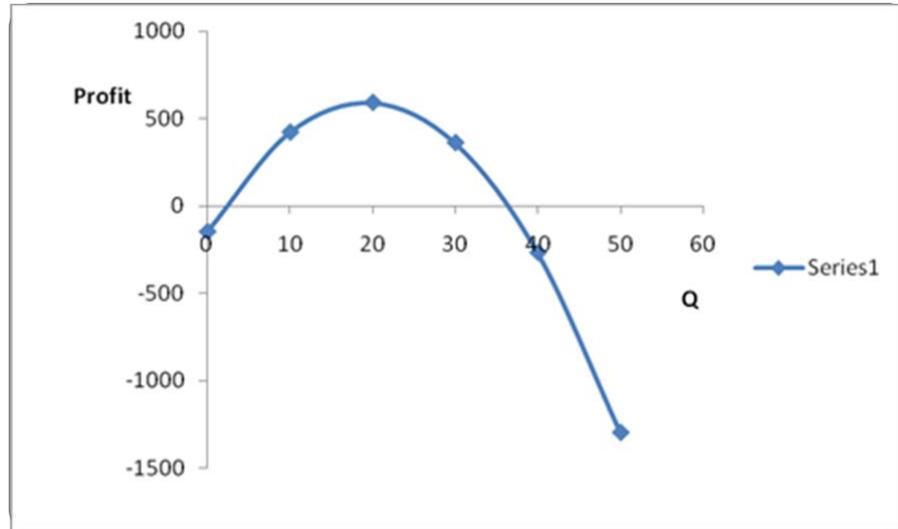


Figure 35 The profit function as a function of quantity produced

3. Determine the break-even points.

Break-even is where Income = Costs.

Look at the graph displaying both Income and Costs and identify the two points where the graphs intersect. This will be where

$$80Q - 2Q^2 = 3Q + 150 \text{ or } 2Q^2 - 77Q + 150 = 0$$

(Note that this equation states that the Profit $P = 0$ at break-even.)

Solve this using the formulas in the “Quadratics and their graphs” section:

$$Q = \frac{-b - \sqrt{\Delta}}{2a} \text{ and } Q = \frac{-b + \sqrt{\Delta}}{2a} \text{ give the two solutions for } Q$$

where $\Delta = b^2 - 4ac$

In this case, $a = 2$, $b = -77$, $c = 150$ and

$$\Delta = (-77)^2 - 4(2)(150) = 4,729$$

$$\text{So } Q = \frac{77 - \sqrt{4,729}}{2(2)} \text{ and } Q = \frac{77 + \sqrt{4,729}}{2(2)} \text{ are solutions.}$$

$$\text{Break-even points are } Q = \frac{77 - 68.77}{4} = 2.06 \text{ and}$$

$$Q = \frac{77 + 68.77}{4} = 36.44$$

Round off the answers: $Q = 2$ bags of rice weekly or 36 bags weekly, for break-even.

If they produce fewer than two bags of rice per week or more than 36 bags per week, the costs will be more than the income. There are different reasons for this – can you name some?

For $2 < Q < 36$, it is clear from the graph that Income exceeds Costs and this is the area where a profit can be made.

4. Calculate the maximum profit and the number of bags and kilograms that should be produced to obtain this.

Look at the Profit graph in Figure 33. Profits increase and then decrease as the number of bags of rice per week increases. It looks as though $Q = 20$ bags may maximise the profit.

From calculus, the maximum is reached where $\frac{dP}{dQ} = 0$

Since $P = -2Q^2 + 77Q - 150$, we see $\frac{dP}{dQ} = -4Q + 77$

$\frac{dP}{dQ} = 0$ implies that $-4Q + 77 = 0$. Solve for Q :

$$4Q = 77 \text{ and } Q = 19.25 \approx 19$$

Therefore, producing 19 bags of rice per week will yield the maximum profit. This is $19 \times 50 = 950$ kg of rice.

The maximum profit is $P = -2(19)^2 + 77(19) - 150 = 591$ dollars per week.

Marginal analysis

It is often useful to look at **marginal costs**. The marginal cost (MC) of manufacturing items is the additional cost of making one extra unit of the item at some point in the production process. It is the gradient of the total cost curve. So if the total cost is y and x denotes number of items

manufactured, then the marginal cost is $\frac{dy}{dx}$

Here is an example:

Your plant in India has produced 1,000 items. It is known that

$MC = \frac{dy}{dx} = 2x - 3$ at this stage. What would be the total cost for an

additional 1,000 items to be manufactured? Can you afford it?

You need to calculate the cost $y(x)$ for $1,000 \leq x \leq 2,000$. This is given by

the definite integral $\int_{1,000}^{2,000} (2x - 3)dx$

$$\int_{1,000}^{2,000} (2x - 3)dx$$

$$= [x^2 - 3x + K](\text{for } x = 2,000) - [x^2 - 3x + K](\text{for } x = 1,000)$$

$$= [(2,000)^2 - 3(2,000)] - [(1,000)^2 - 3(1,000)]$$

$$= 997,000$$

It would cost INR 997,000 to manufacture another 1,000 items.

Remember these key points

- $\frac{dy}{dx}$ means the rate of change of quantity y with respect to x and is called the derivative of y with respect to x . Graphically, it gives the slope of the graph of y . For any constant quantity c , we have $\frac{dc}{dx} = 0$. For any constant a and integer n , we have $\frac{d}{dx} a x^n = a n x^{n-1}$
- A function y has a turning point where $\frac{dy}{dx} = 0$. In our applications, this will be either a minimum or a maximum. The test for a minimum is that $\frac{d^2y}{dx^2} > 0$ and the test for a maximum is that $\frac{d^2y}{dx^2} < 0$. This enables us to optimise quantities such as profits and costs.
- For quadratic functions, the turning point is at the point $x = \frac{-b}{2a}$
A function is increasing where slope $\frac{dy}{dx} > 0$ and decreasing where slope $\frac{dy}{dx} < 0$
- Integration is the reverse of differentiation: $\int \frac{dy}{dx} dx = y(x) + K$
 $\int x^n dx = \frac{1}{n+1} x^{n+1} + K$, with K any constant.
- Definite integrals $\int_a^b y dx$ give the area between the graph of y and the x -axis between $x = a$ and $x = b$.
- Applications of differentiation include maximisation of profits, minimisation of costs and marginal analysis.



Unit summary

You have successfully completed this unit if you can:



Summary

- **differentiate** functions;
- **calculate** rates of change;
- **find** the maxima and minima of quantities;
- **perform** basic integration; and
- **apply** differentiation and integration in management problems.

Assessment



Assessment – Module 1

- If the amount \$10,500.00 includes tax of 12%, what was the original amount excluding tax?
- What is the rate of return per annum on an investment with value that increases from 12,000 to 13,634 over the year?
- You are given the sequence 2, 3, 4.5, 6.75...
 - Find the formula for the general term of the sequence.
 - Determine the sum of the first 20 terms of the series.
- An investment of 10,000 cedi has a cash inflow of 4,500 cedi at the end of each year for the next three years. The discount rate is 4.5%.
 - Calculate the present value of all cash flows.
 - Determine the *NPV* and *PI* for the investment. Discuss your findings.
 - Do a sensitivity analysis for a scenario where you expect the interest rate to go up.
- Draw the graph and give the equation for the linear relationship between quantities *P* and *Q* if you are given the two data points (*P*, *Q*): (22, 4) and (46, 12). Take *Q* as the dependent variable.

- Solve these simultaneous equations both graphically and algebraically:

$$2y - 2x + 6 = 0 \text{ and } 2y = x - 5$$

Now simultaneously solve the inequalities $2y - 2x + 6 \geq 0$ and $2y \geq x - 5$

- A cell phone stall allows customers to make calls of three minutes' duration. Let *Q* denote the average number of phone calls made per week. Costs are denoted in units of dollars.

The total weekly cost function for supplying the service is
 $C = 3Q + 50$

The total weekly income function for this service is
 $I = 100Q - 0.5Q^2$

Analyse the situation for break-even and profit, following the method in Unit 4.

- Find the maximum value of $f(x)$ if $f(x) = -3x^2 + 5x - 1$
- Determine the value of the definite integral and interpret it:

$$\int_{-2}^2 (-x^2 + 4) dx$$

- Let the total cost of production be *y* and let *x* denote the number of items manufactured. The marginal cost is $MC = \frac{dy}{dx}$



A small factory in Malaysia has produced 200 items. It would like to expand its operations and produce 300 items, but only has 20,000 rupees available to fund expansion. It is known that marginal cost

$$MC = \frac{dy}{dx} = x - 4$$
 at this stage. What would be the total cost for the

next 100 items to be manufactured? Can the factory afford it?

References



References

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Further reading



Reading

Where else can I look?

These texts and websites may be consulted for additional information:

Black, J. & Bradley, J. F. (1973). *Essential mathematics for economists*. London, England: John Wiley & Sons.

Bradley, T. & Patton, P. (2002). *Essential mathematics for economics and business*. New York: John Wiley & Sons.

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